

Some Common Fixed Point Theorem in Complete Metric Space of Integral Type

¹Jagdish C. Chaudhary, ²Shailesh T. Patel

¹Research Scholar & Faculty of Science Pacific Academy of Higher Education and Research University, Udaipur, India

²S.P.B. Patel Engg. College, Linch, Mehsana, Gujarat, India

Abstract—

In this paper, we prove some common fixed point theorems in complete metric spaces for self mapping satisfying a contractive condition of Integral type.

Keywords— Continuous Mapping, Compatible Mappings, Fixed point, Common Fixed point and Complete metric space.

I. INTRODUCTION

The first result on fixed points for contractive type mapping was the Banach's contraction principle by S. Banach [1] in 1922. This theorem runs as the following,

Theorem 1.1 (Banach's contraction principle) Let (X, d) be a complete metric space, $k \in (0, 1)$ and $f: X \rightarrow X$ be a mapping such that for each $x, y \in X$, $d(f_x, f_y) \leq kd(x, y)$ 1.1

Then f has a unique fixed point $c \in X$, such that for each $x \in X$, $\lim_{n \rightarrow \infty} f^n(x) = c$. After the classical result, Kannan [5] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions.

In 2002, A. Branciari [2] analyzed the existence of fixed points for mapping f defined on a complete metric space X , d satisfying a general contractive condition of integral type.

Theorem 1.2(Branciari) Let (X, d) be a complete metric space, $k \in (0, 1)$ and let $f: X \rightarrow X$ be a mapping such that for each $x, y \in X$, $\int_0^{d(f_x, f_y)} \xi(t) dt \leq k \int_0^{d(x, y)} \xi(t) dt$ 1.2

Where $\xi: [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$, non negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \xi(t) dt > 0$, then f has a unique fixed point $c \in X$ such that for each $x \in X$, $\lim_{n \rightarrow \infty} f^n(x) = c$.

After the paper of Branciari, a lot of a research works have been carried out on generalizing contractive conditions of integral type for a different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [6] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(f_x, f_y)} \xi(t) dt \leq \int_0^{\max\{d(x, y), d(x, f_x), d(y, f_y), \frac{d(x, f_y) + d(y, f_x)}{2}\}} \xi(t) dt$$
1.3

The aim of this paper is to generalize some mixed type of contractive conditions to the mapping and then a pair of mappings, satisfying a general contractive mappings such as Kannan type [3], Chatrterjee type [5], Zamfirescu type [11], etc.

II. MAIN RESULTS

Theorem 2.1 Let f be a self mapping of a complete metric space (X, d) , Satisfying the following condition:

$$\int_0^{d(f_x, f_y)} \xi(t) dt \leq \alpha \int_0^{[d(x, f_x) + d(y, f_y)]} \xi(t) dt + \beta \int_0^{[d(x, f_y) + d(y, f_x)]} \xi(t) dt + \gamma \int_0^{\max\{d(x, f_y), d(y, f_x), d(x, f_x), d(y, f_y)\}} \xi(t) dt + \delta \int_0^{d(x, y)} \xi(t) dt$$
2.1

For each $x, y \in X$ with non negative reals $\alpha, \beta, \gamma, \delta$ such that $0 < 2\alpha + 2\beta + 2\gamma + \delta < 1$, where $\xi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a lebesgue- integrable mapping which is summable on each compact subset of \mathbb{R}^+ , non negative and such that for each $\epsilon > 0$, $\int_0^\epsilon \xi(t) dt > 0$ 2.2

Then f has a unique fixed point $z \in X$ and for each $x \in X$, $\lim_{n \rightarrow \infty} f^n x = z$

Proof: For any arbitrary $x_0 \in X$, we define a sequence $\{x_n\}$ of element of X , such that $x_{n+1} = f x_n$, for $n = 0, 1, 2, 3 \dots$

$$\text{Now } \int_0^{d(x_{n+1}, x_{n+2})} \xi(t) dt = \int_0^{d(fx_n, fx_{n+1})} \xi(t) dt$$

Form 2.1

$$\begin{aligned} & \int_0^{d(fx_n, fx_{n+1})} \xi(t) dt \leq \alpha \\ & \int_0^{[d(x_n, fx_n) + d(x_{n+1}, fx_{n+1})]} \xi(t) dt + \\ & \beta \int_0^{[d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)]} \xi(t) dt + \\ & \quad \gamma \int_0^{\max [d(x_n, fx_{n+1}), d(x_{n+1}, fx_n), d(x_n, fx_n), d(x_{n+1}, fx_{n+1})]} \xi(t) dt + \\ & \delta \int_0^{d(x_n, x_{n+1})} \xi(t) dt \\ & \int_0^{d(x_{n+1}, x_{n+2})} \xi(t) dt \leq \alpha \\ & \int_0^{[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]} \xi(t) dt + \\ & \beta \int_0^{[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})]} \xi(t) dt + \\ & \quad \gamma \int_0^{\max [d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})]} \xi(t) dt + \\ & \delta \int_0^{d(x_n, x_{n+1})} \xi(t) dt \end{aligned}$$

Which implies,

$$\int_0^{d(x_{n+1}, x_{n+2})} \xi(t) dt \leq \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right) \int_0^{d(x_n, x_{n+1})} \xi(t) dt \tag{2.3}$$

Thus, by routine calculation

$$\int_0^{d(x_n, x_{n+1})} \xi(t) dt \leq m^n \int_0^{d(x_0, x_1)} \xi(t) dt \tag{2.4}$$

Where $m = \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right) < 1$

Taking limit of (2.4) as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \int_0^{d(x_n, x_{n+1})} \xi(t) dt = 0 \tag{2.5}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p) + 1$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \epsilon, \quad d(x_{m(p)}, x_{n(p)-1}) \geq \epsilon \tag{2.6}$$

Now

$$\begin{aligned} & d(x_{m(p)-1}, x_{n(p)-1}) < \\ & d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) \\ & d(x_{m(p)-1}, x_{n(p)-1}) < d(x_{m(p)-1}, x_{m(p)}) + \epsilon \end{aligned} \tag{2.7}$$

Hence

$$\lim_{n \rightarrow \infty} \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \xi(t) dt = \int_0^\epsilon \xi(t) dt \tag{2.8}$$

Using (2.3), (2.6), and (2.8), we get

$$\begin{aligned} & \int_0^\epsilon \xi(t) dt \leq \int_0^{d(x_{m(p)}, x_{n(p)})} \xi(t) dt \leq \\ & m \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \xi(t) dt \leq m \int_0^\epsilon \xi(t) dt \end{aligned}$$

Which is contradiction,

Since $\epsilon \in (0, 1)$, therefore $\{x_n\}$ is a Cauchy, hence converges to $z \in X$

From 2.1, we get

$$\begin{aligned} & \int_0^{d(fz, fz)} \xi(t) dt \leq \alpha \int_0^{[d(z, fz) + d(x_n, fx_n)]} \xi(t) dt + \\ & \beta \int_0^{[d(z, fx_n) + d(x_n, fz)]} \xi(t) dt + \\ & \quad \gamma \int_0^{\max [d(z, fx_n), d(x_n, fz), d(z, fz), d(x_n, fx_n)]} \xi(t) dt + \\ & \delta \int_0^{d(z, x_n)} \xi(t) dt \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get $\int_0^{d(fz, z)} \xi(t) dt \leq (2\alpha + 2\beta + 2\gamma + \delta) \int_0^{d(fz, z)} \xi(t) dt$ as $2\alpha + 2\beta + 2\gamma + \delta < 1$,

$$\int_0^{d(fz, z)} \xi(t) dt = 0$$

Which from (2.2) implies that $d(z, w) = 0$, so the fixed point is unique.

Remark:

- i) On setting $\xi(t) = 1$ over \mathbb{R}^+ , the contractive condition of integral type transforms into a general contractive condition not involving integrals
- ii) From condition 2.1, of integral type, several contractive mappings of integral type can be obtained.
 - a) $\beta = \gamma = \delta = 0$ and $\alpha \in (0, \frac{1}{2})$ gives Kannan mapping of integral type.
 - b) $\alpha = \beta = \delta = 0$ and $\gamma \in (0, \frac{1}{2})$, gives Chatterjea map of integral type.
 - c) $\beta = 0, \alpha, \gamma \in (0, \frac{1}{2}), \delta \in (0, 1)$, at least one of the following condition hold

$$Z1 : \int_0^{d(f_x, f_y)} \xi(t) dt \leq \alpha \int_0^{[d(x, f_x) + d(y, f_y)]} \xi(t) dt$$

$$Z2 : \int_0^{d(f_x, f_y)} \xi(t) dt \leq \gamma \int_0^{[d(x, f_y) + d(x, f_y)]} \xi(t) dt$$

$$Z3 : \int_0^{d(f_x, f_y)} \xi(t) dt \leq \delta \int_0^{d(x, y)} \xi(t) dt$$

Gives Zamfirescu mapping of integral type.

Example 2.2 let $X = [0, 1]$ and d be usual metric with $(x, y) = |x - y|$. Clearly (X, d) is a complete metric space Let $f: X \rightarrow Y$ be given by $f(x) = \frac{x}{2}$ for all $x \in [0, 1]$. Again $\xi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be given by $\xi(t) = \frac{t}{2}$. For all \mathbb{R}^+ then for each

$$\varepsilon > 0 \quad \int_0^\varepsilon \xi(t) dt = \int_0^\varepsilon \frac{t}{2} dt = \frac{\varepsilon^2}{6} > 0$$

Now taking $\alpha = \beta = \gamma = \frac{1}{16}$ and $\delta = \frac{1}{8}$, one can easily verify that the condition 2.1, of theorem 2.1 is satisfied with $2\alpha + 2\beta + 2\gamma + \delta < 1$ and 0 is of course, the unique fixed point of f .

Now our next theorem is the extension of the theorem 2.1, for a pair of mappings .

Theorem: 2.3 Let f and g be self mapping of a complete metric space (X, d) , satisfying the following condition:

$$\int_0^{d(f_x, g_y)} \xi(t) dt \leq \alpha \int_0^{[d(x, f_x) + d(y, g_y)]} \xi(t) dt +$$

$$\beta \int_0^{[d(x, g_y) + d(y, f_x)]} \xi(t) dt +$$

$$\gamma \int_0^{\max[d(x, g_y), d(y, f_x), d(x, f_x), d(y, g_y)]} \xi(t) dt +$$

$$\delta \int_0^{d(x, y)} \xi(t) dt$$

2.9

For each $x, y \in X$ with non negative reals $\alpha, \beta, \gamma, \delta$ such that $0 < 2\alpha + 2\beta + 2\gamma + \delta < 1$, where $\xi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of \mathbb{R}^+ , non negative and such that for each

$$\varepsilon > 0, \int_0^\varepsilon \xi(t) dt$$

2.10

Then f and g has a unique fixed point $z \in X$ and for each $x \in X, \lim_{n \rightarrow \infty} f^n x = z$

Proof: For any arbitrary $x_0 \in X$, we define a sequence $\{x_n\}$ of element of X , such that $x_{n+1} = f x_n$, and $x_{n+2} = g x_{n+1}$, for $n = 0, 1, 2, 3, \dots$

Now

$$\int_0^{d(x_{n+1}, x_{n+2})} \xi(t) dt = \int_0^{d(f x_n, g x_{n+1})} \xi(t) dt$$

From 2.9

$$\int_0^{d(f x_n, g x_{n+1})} \xi(t) dt \leq \alpha$$

$$\int_0^{[d(x_n, f x_n) + d(x_{n+1}, g x_{n+1})]} \xi(t) dt +$$

$$\beta \int_0^{[d(x_n, g x_{n+1}) + d(x_{n+1}, f x_n)]} \xi(t) dt +$$

$$\gamma \int_0^{\max[d(x_n, g x_{n+1}), d(x_{n+1}, f x_n), d(x_n, f x_n), d(x_{n+1}, g x_{n+1})]} \xi(t) dt +$$

$$\delta \int_0^{d(x_n, x_{n+1})} \xi(t) dt$$

$$\int_0^{d(x_{n+1}, x_{n+2})} \xi(t) dt \leq \alpha$$

$$\int_0^{[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]} \xi(t) dt +$$

$$\beta \int_0^{[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})]} \xi(t) dt +$$

$$\gamma \int_0^{\max[d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})]} \xi(t) dt +$$

$$\delta \int_0^{d(x_n, x_{n+1})} \xi(t) dt$$

Which implies

$$\int_0^{d(x_{n+1}, x_{n+2})} \xi(t) dt \leq \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right) \int_0^{d(x_n, x_{n+1})} \xi(t) dt \quad 2.11$$

Thus by routine calculation

$$\int_0^{d(x_n, x_{n+1})} \xi(t) dt \leq m^n \int_0^{d(x_0, x_1)} \xi(t) dt \quad 2.12$$

Where $m = \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right) < 1$

Taking limit of (2.12) as $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \int_0^{d(x_n, x_{n+1})} \xi(t) dt = 0 \quad 2.13$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p + 1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \epsilon, d(x_{m(p)}, x_{n(p)-1}) \geq \epsilon \quad 2.14$$

Now

$$\begin{aligned} d(x_{m(p)-1}, x_{n(p)-1}) &< \\ d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) &< \\ d(x_{m(p)-1}, x_{n(p)-1}) &< d(x_{m(p)-1}, x_{m(p)}) + \epsilon \end{aligned} \quad 2.15$$

We get

$$\int_0^\epsilon \xi(t) dt \leq \int_0^{d(x_{m(p)}, x_{n(p)})} \xi(t) dt \leq h \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \xi(t) dt \leq h \int_0^\epsilon \xi(t) dt$$

Which is contradiction,

Since $h \in (0,1)$, therefore $\{x_n\}$ is a Cauchy, hence converges to

From 2.9, we get

$$\begin{aligned} \int_0^{d(fz, gx_n)} \xi(t) dt &\leq \alpha \int_0^{[d(z, fz) + d(x_n, gx_n)]} \xi(t) dt + \\ \beta \int_0^{[d(z, gx_n) + d(x_n, fz)]} \xi(t) dt + \\ \gamma \int_0^{\max\{d(z, gx_n), d(x_n, fz), d(z, fz), d(x_n, gx_n)\}} \xi(t) dt + \\ \delta \int_0^{d(z, x_n)} \xi(t) dt \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\begin{aligned} \int_0^{d(fz, z)} \xi(t) dt &\leq (2\alpha + 2\beta + 2\gamma + \delta) \int_0^{d(fz, z)} \xi(t) dt \quad \text{as } 2\alpha + 2\beta + 2\gamma + \delta < 1 \\ \int_0^{d(fz, z)} \xi(t) dt &= 0 \end{aligned}$$

Which from (2.10) implies that $d(fz, z) = 0$ or $fz = z$ similarly it can be shown that $gz = z$ so f and g have a common fixed point $z \in X$. now we show that z is the unique common fixed point of f and g .

If possible suppose that, let w be another common fixed point of f and g . then from 2.9 we have

$$\int_0^{d(w, z)} \xi(t) dt \leq (2\beta + \gamma + \delta) \int_0^{[d(w, gz) + d(z, fw)]} \xi(t) dt$$

Since $(2\beta + \gamma + \delta) < 1$, this implies that $\int_0^{d(w, z)} \xi(t) dt = 0$

Which is from 2.10 implies that $d(z, w) = 0$, and so the fixed point is unique.

Remark: In this theorem 2.3 if we take $f=g$ then we get result of theorem 2.1.

REFERENCES

- [1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux quations intgrales, Fund. Math.3,(1922)133181 (French).
- [2] A. Branciari, A _xed point theorem for mappings satisfying a general contractive condition of integral type, Int.J.Math.Math.Sci, 29(2002), no.9, 531 - 536.
- [3] S.K.Chatterjea, Fixed point theorems, C.R.Acad.Bulgare Sci. 25(1972), 727-730.
- [4] K. Goebel and W. A. Kirk, Topiqs in Metric Fixed Point Theory, Combridge University Press, New York, 1990.
- [5] R. Kannan, Some results on _xed points, Bull. Calcutta Math. Soc. , 60(1968), 71-76.

- [6] B.E . Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type, *International Journal of Mathematics and Mathematical Sciences*, 63, (2003), 4007 - 4013.
- [7] B. E. Rhoades, A Comparison of Various Definitions of Contractive Mappings, *Trans. Amer. Math. Soc.* 226 (1977), 257-290.
- [8] B. E. Rhoades, Contractive definitions revisited, *Topological Methods in Nonlinear Functional Analysis* (Toronto, Ont.,1982), *Contemp. Math.*, 21, American Mathematical Society, Rhode Island, (1983), 189-203.
- [9] B. E. Rhoades, Contractive Definitions, *Nonlinear Analysis*, World Science Publishing, Singapore, 1987, 513-526.
- [10] O. R. Smart, *Fixed Point Theorems*, Cambridge University Press, London, 1974.
- [11] T.Zamfirescu, Fixed point theorems in metric spaces, *Arch.Math.(Basel)* 23(1972), 292-298.
- [12] R.Bhardwaj, Some common fixed point theorem in metric space using integral type mappings, *IOSRJEN*,(2012),187-190.