

# Solving Open Queueing Network Problem

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## Abstract:

Queueing networks may be considered as a group of inter-connected nodes (say  $k$ ) where each node represents a service facility of some kind with  $s_i$  servers at the node  $i$  ( $s_i \geq 1$ ). We propose to apply 1<sup>st</sup> the method given by J. R. Jackson and then we would like to solve the same problem by the method of J. D. C. Little.

Keywords- Queueing Networks, Node, J. R. Jackson, J. D. C. Little, Server.

## I. INTRODUCTION

The multiserver queueing network model is denoted by  $M/M/s$ . It is used for analyzing service stations with more than one server. The arrival of customers is assumed to follow a Poisson process and service times assumed to be exponentially distributed.

In Markovian queueing network, each node represents an  $M/M/s$  queue with  $s_i$  servers at node  $i$  ( $i = 1, 2, 3, \dots, k$ ) and there are free transitions among the nodes. As a result of which each queue of the system becomes a waiting room of infinite size. Further suppose that the customers arrive at the node  $i$  from outside the network in a Poisson process with rate  $\lambda_i$  and that service times at node  $i$  are exponential with mean  $\frac{1}{\mu_i}$ .

## II. MATHEMATICAL FORMULATION

Let  $P_{ij}$  be the probability that a customer completing service at node  $i$  request service from node  $j$  where  $j \neq i$  and let  $P_{i0}$  be the probability that it will leave the network after service at node  $i$ . Let  $Q_1, Q_2, \dots, Q_k$  be the number of customers with  $k$  nodes respectively, as  $t \rightarrow \infty$  where

$$p_{n_1 n_2 \dots n_k} = P(Q_1 = n_1, Q_2 = n_2, \dots, Q_k = n_k).$$

This is an example of open Jackson network who has the credit of analyzing it for the first time. For the limiting distribution  $p_{n_1 n_2 \dots n_k}$ , Jackson has shown that

$$p_{n_1 n_2 \dots n_k} = p_1(n_1) p_2(n_2) \dots p_k(n_k)$$

Where

$$p_i(r) = \begin{cases} p_i(0) \frac{(\gamma_i/\mu_i)^r}{r!}, & r = 0, 1, 2, \dots, s_i, \\ p_i(0) \frac{(\gamma_i/\mu_i)^r}{s_i! s_i^{r-s_i}}, & r = s_i, s_i + 1, \dots, \end{cases} \dots (1)$$

and

$$\gamma_i = \lambda_i + \sum_{j=1}^k P_{ji} \gamma_j, \quad i = 1, 2, \dots, k \dots (2)$$

It is to be noted that  $\gamma_i$  can be determined from equation (2) when  $\lambda_i$  and  $P_{ij}$  ( $i, j = 1, 2, \dots, k$ ) are nodes here.  $\gamma_i$  is the effective arrival rate at node  $i$  after taking into account the traffic from outside the network and  $k-1$  other nodes within the network. Thus if  $\rho_i = \gamma_i/\mu_i$  is the effective traffic intensity at node, then  $\rho_i < 1$  where  $i = 1, 2, \dots, k$  for the limiting distribution to exist. Now  $p_i(0)$  for  $i = 1, 2, \dots, k$  can be determined using the normalizing condition

$$\sum_{n_1} \sum_{n_2} \dots \sum_{n_k} p_{n_1 n_2 \dots n_k} = 1.$$

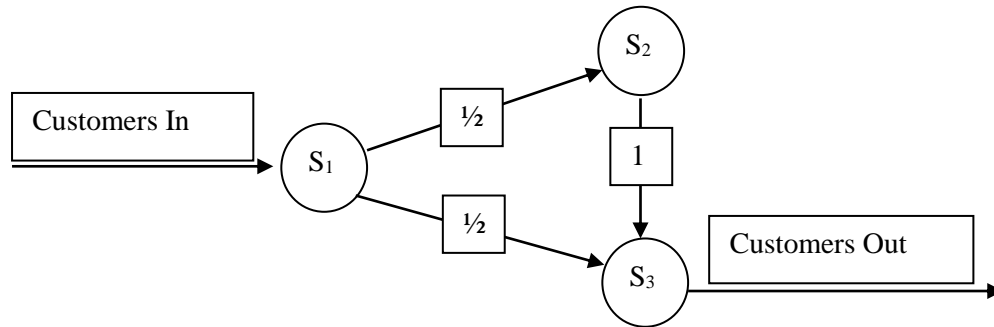
The structure of the distribution  $p_i(r)$  in (1) is similar to the limiting distribution of the queue  $M/M/s_i$  with arrival rate  $\gamma_i$  and service rate  $\mu_i$ .

We are giving below an illustrative example for solution of an open Jackson network problem that obeys Poisson process.

In a network stations  $S_1, S_2, S_3$ , customers arrive at  $S_1, S_2, S_3$ , from outside in accordance with Poisson process having rates 30, 40, 50 respectively. The service times at 3 stations are exponentially distributed with rates 40, 100, 150

respectively. A customer completing the service is equally likely to go to  $S_2$  or to  $S_3$ . A customer departing from service station  $S_2$  always goes to  $S_3$  from where he leaves the system.

- (i) Calculate average number of customers in the system.
- (ii) Also find the average waiting time of a customer in the system.



**The system given above is a Jackson's open queue system.**

Let  $\gamma_1, \gamma_2, \gamma_3$ , be the resultant arrival rates at  $S_1, S_2$  and  $S_3$  respectively. Jackson's flow balance equations are

$$\gamma_i = \lambda_i + \sum_{j=1}^3 P_{ji} \gamma_j, \quad i = 1, 2, 3 \quad \dots (3)$$

We note that  $P_{12} = \frac{1}{2}, P_{13} = \frac{1}{2}, P_{23} = 1$

Putting  $i=1$  in equation (3), we get

$$\gamma_1 = \lambda_1 + P_{11}\gamma_1 + P_{21}\gamma_2 + P_{31}\gamma_3$$

(since  $P_{11} = P_{21} = P_{31} = 0$ )

i.e.,  $\gamma_1 = 30 \dots \dots \dots (4)$

Putting  $i=2$  in equation (3), we get

$$\gamma_2 = \lambda_2 + P_{12}\gamma_1 + P_{22}\gamma_2 + P_{32}\gamma_3$$

i.e.,  $\gamma_2 = 40 + \frac{1}{2} \times 30$  (since  $P_{22} = P_{32} = 0$ ) (by using (4))

i.e.,  $\gamma_2 = 55 \dots \dots \dots (5)$

Putting  $i=3$  in equation (3), we get

$$\gamma_3 = \lambda_3 + P_{13}\gamma_1 + P_{23}\gamma_2 + P_{33}\gamma_3$$

i.e.,  $\gamma_3 = 50 + \frac{1}{2} \times 30 + 1 \times 55$  (since  $P_{33} = 0$ ) (by using (4) & (5))

i.e.,  $\gamma_3 = 120$

- (i) Thus, average number of customers in the system,

$$\begin{aligned}
 E(N_s) &= \frac{\gamma_1}{\mu_1 - \gamma_1} + \frac{\gamma_2}{\mu_2 - \gamma_2} + \frac{\gamma_3}{\mu_3 - \gamma_3} \\
 &= \frac{30}{40 - 30} + \frac{55}{100 - 55} + \frac{120}{150 - 120} \\
 &= 3 + \frac{11}{9} + 4 = \frac{74}{9} \\
 &= 8.2222
 \end{aligned}$$

- (ii) And the average waiting time of one customer in the system,

$$\begin{aligned}
 E(W_s) &= \frac{E(N_s)}{\gamma}, \quad \text{where } \gamma = \lambda_1 + \lambda_2 + \lambda_3 = 120 \\
 &= \frac{8.2222}{120} = .0685
 \end{aligned}$$

**We now proceed to examine the above results using Little's law named after J.D.C. Little.**

$$E(N_s) = E(N_q) + \frac{\gamma}{\mu}$$

Here,

$E(N_s)$  = Average number of customers in the system

$E(N_q)$  = Average number of customers in the queue

Where  $\gamma = \lambda_1 + \lambda_2 + \lambda_3 = 120$  and  $\mu = \mu_1 + \mu_2 + \mu_3 = 290$

$$E(N_s) = \frac{1}{s \cdot s!} \cdot \frac{\left(\frac{\gamma}{\mu}\right)^{s+1}}{\left(1 - \frac{\gamma}{\mu s}\right)^2} P_0 + \frac{\gamma}{\mu}$$

$$P_0 = \frac{1}{\left\{ \sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\gamma}{\mu}\right)^n \right\} + \left\{ \frac{1}{s!} \left(1 - \frac{\gamma}{\mu s}\right) \left(\frac{\gamma}{\mu}\right)^s \right\}}$$

$$= \frac{1}{\left\{ 1 + .4137 + \frac{1}{2} \times (.4137)^2 \right\} + \left\{ \frac{1}{3!} \left(1 - \frac{120}{290 \times 3}\right) \times \left(\frac{120}{290}\right)^3 \right\}}$$

$$= \frac{1}{\{1 + .4137 + .0855\} + \left\{ \frac{1}{6 \left(\frac{25}{29}\right)} \times (.4137)^3 \right\}}$$

$$= \frac{1}{1 + .4137 + .0855 + \frac{.0708}{5.1724}}$$

$$= \frac{1}{1.4137 + .0855 + .0136}$$

$$= .6610$$

(i) Thus, average number of customers in the system,

$$E(N_s) = \frac{1}{s \cdot s!} \cdot \frac{\left(\frac{\gamma}{\mu}\right)^{s+1}}{\left(1 - \frac{\gamma}{\mu s}\right)^2} P_0 + \frac{\gamma}{\mu} \dots \dots \dots (6)$$

Now putting the value of  $P_0$  in equation (6)

We get:

$$= \frac{1}{3 \times 6} \cdot \frac{(.4137)^4}{\left(1 - \frac{.4137}{3}\right)^2} \times .6610 + .4137$$

$$= \frac{.0292}{18 \times (.8621)^2} \times .6610 + .4137$$

$$= \frac{.0193}{13.3779} + .4137$$

$$= .4151$$

(ii) And the average waiting time of one customer in the system,

$$E(W_s) = \frac{1}{\gamma} E(N_s)$$

$$= \frac{1}{120} \times .4151$$

$$= .0034$$

### III. CONCLUSIONS

On the basis of the results obtained Jackson's method seems to be more efficient in comparison to the method given by J. D. C. Little. Open Queueing network problems of  $M/M/s$  model have practical importance and thus study of this model is significant.

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