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A Common Fixed Point Theorem under Reciprocal Continuity

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Abstract:

In a paper of R.P. Pant[10] the first theorem guaranteeing the existence of a common fixed point even when all the mappings may be discontinuous and some of the mappings may not be satisfying the compatibility . In the present paper we obtain a common fixed point theorem by using reciprocal continuity condition . Our theorem generalizes a multitude of common fixed point theorems.

Key Words: Compatible Maps; Reciprocal Continuity; Contractive Conditions and Common Fixed Points. AMS Subject classification (2000); primary 54H25, secondary 47H10.

I. INTRODUCTION

Jungck et. al [4] proved some interesting common fixed point theorems for contractive type mappings . Jachymski [2] and Pant [6,7] also proved the same result . These theorem invariably require a compatibility condition and a contractive condition besides assuming continuity of atleast one of the mappings and each theorem aims at weakening one or more of these conditions . These theorems require one of the mappings in a compatible pair to be continuous . For example , Theorem 3.3 of Jachymski [2] assumes one of the mappings to be continuous while Theorem 5.1 of the same paper assumes S or T to be continuous . The main theorem of Rhoades et al. [9] and Jungck et al. [4] requires S or T to be continuous . Likewise, the theorems of Fisher [1] and Pant [5,6,7] assume one of the mappings to be continuous . However, Pant [10] obtained a common fixed point theorem by using fewer compatibility and contractive conditions and by using reciprocal continuity condition . Pant [10] also proved that in the setting of common fixed point theorems for compatible maps satisfying contractive conditions , the notion of reciprocal continuity is weaker than the assumption of continuity of one of the mappings.

II. PRELIMINARIES

Definition 1. Two selfmaps A and S of a metric space (X,d) are called compatible if

$$\lim_{n\to\infty} d(ASx_n,SAx_n) = 0$$

whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some } t \text{ in } X.$$

This notion is due to Jungck [3] and implies commutativity at coincidence points.

Definition 2. Two selfmaps A and S of a metric space (X,d) will be called reciprocally continuous if

$$\lim_{n\to\infty} ASx_n = At \quad \text{and} \quad \lim_{n\to\infty} SAx_n = St$$

whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some } t \text{ in } X .$$

If A and S are both continuous then they are obviously reciprocally continuous but the converse is not true . Moreover ,Pant [10] proved that in the setting of common fixed point theorems for compatible maps satisfying contractive conditions , continuity of one of the mappings A or S implies their reciprocal continuity but not conversely.

III. MAIN RESULTS

In this section we gave a generalized result of Pant [10] by using a functional inequality.

Theorem . Let $\{A_i\}$, i = 1,2,3, . . . , S and T be selfmappings of a complete metric space (X,d) such that for $\alpha ,\beta >0$ with $\alpha + \beta < 1$ and for all x,y in X , we have

(i) $A_1X\subseteq TX$ and $A_iX\subseteq SX$ when i > 1

(ii) $$d(A_1x,A_2y) \le \frac{\alpha\, \phi(d(A_2y,Ty))\, [1 + \phi(d(A_1x,Sx))]}{[1 + \phi(d(Sx,Ty))]} + \beta\, \phi(d(Sx,Ty))$$

(iii) $$d(A_1x,A_iy) < \frac{\alpha\, d(A_iy,Ty)\, [1 + d(A_1x,Sx)]}{[1 + d(Sx,Ty)]} + \beta\, d(Sx,Ty)$$

where $\phi : R_+\rightarrow R_+$ denote an upper semicontinuous function such that $\phi(t) < t$ for each $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$. Let S be compatible with A_1 and T be compatible with A_k for some $k > 1$. If the mappings in one of the compatible pairs (A_1,S) or (A_k,T) be reciprocally continuous then all the A_i , S and T have a unique common fixed point .

Proof. Let x_0 be any point in X . From (i), we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follow $y_{2n} = A_1x_{2n} = Tx_{2n+1}$, $y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}$.

Then using (ii) we obtain

$d(y_{2n},y_{2n+1}) = d(A_1x_{2n},A_2x_{2n+1})$

$$d(y_{2n},y_{2n+1}) \le \frac{\alpha\, \phi(d(A_2x_{2n+1},Tx_{2n+1}))\, [1 + \phi(d(A_1x_{2n},Sx_{2n}))]}{[1 + \phi(d(Sx_{2n},Tx_{2n+1}))]} + \beta\, \phi(d(Sx_{2n},Tx_{2n+1}))$$

$$d(y_{2n},y_{2n+1}) \le \frac{\alpha\, \phi(d(y_{2n+1},y_{2n}))\, [1 + \phi(d(y_{2n},y_{2n-1}))]}{[1 + \phi(d(y_{2n-1},y_{2n}))]} + \beta\, \phi(d(y_{2n-1},y_{2n}))$$

$d(y_{2n},y_{2n+1}) \le \alpha\, \phi(d(y_{2n+1},y_{2n})) + \beta\, \phi(d(y_{2n-1},y_{2n}))$ . . . (1)

Now we have given that $\phi(t) < t$ for $t > 0$ and so we obtain

$d(y_{2n},y_{2n+1}) < \alpha\, d(y_{2n+1},y_{2n}) + \beta\, d(y_{2n-1},y_{2n}) \Rightarrow (1 - \alpha)\, d(y_{2n},y_{2n+1}) < \beta\, d(y_{2n-1},y_{2n})$

$d(y_{2n},y_{2n+1}) < (\beta/(1 - \alpha))\, d(y_{2n-1},y_{2n}) < d(y_{2n-1},y_{2n})$

i.e. $d(y_{2n},y_{2n+1}) < d(y_{2n-1},y_{2n})$ . . . (2)

We thus see that $\{d(y_n,y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence tends to a limit $r \ge 0$. Suppose $r > 0$. Then the inequality (1) on making $n \to \infty$ and in view of upper semicontinuity of ϕ yields $r < \phi(r) < r$, a contradiction .

Hence , $r = \lim_{n \to \infty} d(y_n,y_{n+1}) = 0$ . . . . (3)

We show that $\{y_n\}$ is a cauchy sequence. Suppose it is not. Then there exists $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i},y_{n_{i+1}}) > 2\varepsilon$. Since $\lim_{n \to \infty} d(y_n,y_{n+1}) = 0$,

there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i},y_{m_i}) \ge \varepsilon$. If not, then

$d(y_{n_i},y_{n_{i+1}}) \le d(y_{n_i},y_{n_{i+1}-1}) + d(y_{n_{i+1}-1},y_{n_{i+1}})$

$d(y_{n_i},y_{n_{i+1}}) < \varepsilon + d(y_{n_{i+1}-1},y_{n_{i+1}}) < 2\varepsilon$,

a contradiction . If m_i be the smallest integer such that $d(y_{n_i},y_{m_i}) > \varepsilon$, then

$\varepsilon < d(y_{n_i},y_{m_i}) \le d(y_{n_i},y_{m_i-2}) + d(y_{m_i-2},y_{m_i-1}) + d(y_{m_i-1},y_{m_i})$

$< \varepsilon + d(y_{m_i-2},y_{m_i-1}) + d(y_{m_i-1},y_{m_i})$

That is , there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i},y_{m_i}) \ge \varepsilon$

and $\lim_{n \to \infty} d(y_{n_i},y_{m_i}) = \varepsilon$ . . . (4)

Moreover , m_i can be chosen in such a manner that m_i is even when n_i is odd and m_i is odd when n_i is even. Suppose that n_i is odd and m_i is even . Then by virtue of (ii),

we get $d(y_{n_i+1},y_{m_i+1}) = d(A_1x_{n_i+1},A_2x_{m_i+1})$

$$\le \frac{\alpha\, \phi(d(A_2x_{m_i+1},Tx_{m_i+1})\, [1 + \phi(d(A_1x_{n_i+1},Sx_{n_i+1}))]}{[1 + \phi(d(Sx_{n_i+1},Tx_{m_i+1}))]} + \beta\, \phi(d(Sx_{n_i+1},Tx_{m_i+1}))$$

$$d(y_{n_i+1},y_{m_i+1}) \le \frac{\alpha\, \phi(d(y_{m_i+1},y_{m_i}))\, [1 + \phi(d(y_{n_i+1},y_{n_i}))]}{[1 + \phi(d(y_{n_i},y_{m_i}))]} + \beta\, \phi(d(y_{n_i},y_{m_i}))$$

On letting $n_i \to \infty$ and in view of (3) , (4) and upper semicontinuity of ϕ , the above inequality yields $\varepsilon < \phi(\varepsilon) < \varepsilon$, a contradiction . Hence , $\{y_n\}$ is a cauchy sequence . Since X is complete , there exists a point z in X such that $y_n \to z$.

Also $y_{2n} = A_1x_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2} \to z$.

We show that $A_ix_{2n+1} \to z$ for each $i > 2$. Suppose on the contrary that $\lim_{n \to \infty} A_ix_{2n+1} \ne z$ for $i > 2$,

then there exists a subsequence $\{A_ix_{2m+1}\}$ of $\{A_ix_{2n+1}\}$, a number $r > 0$ and a positive integer M such that for each $m \ge M$ we have $d(A_1x_{2m},A_ix_{2m+1}) \ge r$, $d(A_ix_{2m+1},z) \ge r$

Now

$$d(A_1x_{2m},A_ix_{2m+1}) \le \frac{\alpha\, d(A_ix_{2m+1},Tx_{2m+1})\, [1 + d(A_1x_{2m},Sx_{2m})]}{[1 + d(Sx_{2m},Tx_{2m+1})]} + \beta\, d(Sx_{2m},Tx_{2m+1})$$

$$d(y_{2m},y_{2m+1}) \le \frac{\alpha\, d(y_{2m+1},y_{2m})\, [1 + d(y_{2m},y_{2m-1})]}{[1 + d(y_{2m-1},y_{2m})]} + \beta\, d(y_{2m-1},y_{2m})$$

$$d(y_{2m},y_{2m+1}) \le \alpha\, d(y_{2m+1},y_{2m}) + \beta\, d(y_{2m-1},y_{2m})$$
$$(1-\alpha)\, d(y_{2m},y_{2m+1}) \le \beta\, d(y_{2m-1},y_{2m})$$
$$d(y_{2m},y_{2m+1}) \le (\beta/(1-\alpha))\, d(y_{2m-1},y_{2m}) < d(y_{2m-1},y_{2m})$$
$$d(y_{2m},y_{2m+1}) < d(y_{2m-1},y_{2m}) \quad \ldots(5)$$

This inequality is similar to inequality (2) . Hence we have

$$\lim_{n\to\infty} d(y_{2m},y_{2m+1}) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} d(A_1x_{2m},A_ix_{2m+1}) = 0$$

i.e. $r \le \lim_{n\to\infty} d(A_1x_{2m},A_ix_{2m+1}) = 0 \;\Rightarrow\; r \le 0$

which contradicts the fact that $r > 0$. Hence our supposition is wrong , therefore , $A_ix_{2n+1} \to z$ for $i > 2$. Suppose that T is compatible with A_k for some $k > 1$ and T and A_k are reciprocally continuous . Then the reciprocal continuity of A_k and T implies that

$$\lim_{n\to\infty} A_kTx_{2n+1} = A_kz \quad \text{and} \quad \lim_{n\to\infty} TA_kx_{2n+1} = Tz .$$

Compatibility of A_k and T yields

$$\lim_{n\to\infty} d(A_kTx_{2n+1},TA_kx_{2n+1}) = 0 ,$$

that is , $d(A_kz,Tz) = 0$. Hence , $A_kz = Tz$.

Since $A_kX \subseteq SX$, there exists a point u in X such that $A_kz = Su$. We show that $Su = A_1u$. If not , then $d(A_1u,Su) > 0$. Now

$$d(A_1u,A_kz) \le \frac{\alpha\, d(A_kz,Tz)\,[1 + d(A_1u,Su)]}{[1 + d(A_1u,Tz)]} + \beta\, d(Su,Tz)$$

$$d(A_1u,A_kz) \le \alpha \,.\, 0 + \beta \,.\, 0 \;\Rightarrow\; d(A1u,Akz) \le 0 \;\Rightarrow\; d(A_1u,Su) \le 0$$

which contradicts the fact that $d(A_1u,Su) > 0$. Hence, $Tz = A_kz = Su = A_1u$.

Compatibility of A_1 and S implies that $A_1Su = SA_1u$ and , hence , $A_1A_1u = A_1Su = SA_1u = SSu$. Similarly , compatibility of A_k and T implies that $A_kA_kz = A_kTz = TA_kz = TTz$. If $A_1u \ne A_1A_1u$, using (iii) , we get

$$d(A_1u,A_1A_1u) = d(A_1A_1u,A_1u)$$
$$d(A_1u,A_1A_1u) = d(A_1A_1u,A_kz)$$
$$\le \frac{\alpha\, d(A_kz,Tz)\,[1 + d(A_1A_1u,SA_1u)]}{[1 + d(SA_1u,Tz)]} + \beta\, d(SA_1u,Tz)$$
$$d(A_1u,A_1A_1u) \le 0 + \beta\, d(A_1A_1u,A_1u) < d(A_1A_1u,A_1u)$$

i.e. $d(A_1u,A_1A_1u) < d(A_1u,A_1A_1u)$,

a contradiction . Hence , $A_1u = A_1A_1u$ and $A_1u = A_1A_1u = SA_1u$, that is , A_1u is a common fixed point of A_1 and S . Similarly , using (iii) we find that A_kz $(=A_1u)$ is a common fixed point of A_k and T . Moreover , if $A_kz \ne A_kA_kz$, for some $k > 1$, using (iii) we get

$$d(A_kz,A_kA_kz) = d(A_1u,A_kA_kz)$$
$$\le \frac{\alpha\, d(A_kA_kz,TA_kz)\,[1 + d(A_1u,Su)]}{[1 + d(Su,TA_kz)]} + \beta\, d(Su,TA_kz)$$
$$d(A_kz,A_kA_kz) \le 0 + \beta\, d(A_kz.A_kA_kz) \;\Rightarrow\; d(A_kz,A_kA_kz) \le \beta\, d(A_kz,A_kA_kz) < d(A_kz,A_kA_kz)$$

i.e. $d(A_kz,A_kA_kz) < d(A_kz,A_kA_kz)$,

a contradiction . Hence , A_kz $(=A_1u)$ is a common fixed point of T and A_k for every $k > 1$. Uniqueness of the common fixed point follows easily . The proof is similar when A_1 and S are assumed reciprocally continuous. This completes the proof of our theorem.

We now give an example to illustrate the above theorem:

Example . Let $X = [1,10]$ and d be the usual metric on X . Define $\{A_i\}$, S and $T : X \to X$ by

$A_1x = 1$ for each x

$Sx = x$ if $x \le 4$, $Sx = 4$ if $4 < x \le 7$ and $Sx = (x + 10)/6$ if $x > 7$

$Tx = 1$ if $x < 2$ or $\ge 5/2$, $Tx = (x + 9)/2$ if $2 \le x < 5/2$

$A_2x = 1$ if $x < 2$ or $\ge 5/2$, $A_2x = (x - 1)$ if $2 \le x < 5/2$

and for $i > 2$, $A_ix = 1$ if $x < 2 + (1/i)$ or $\ge 5/2$, $A_ix = (x - 1)$ if $2 + (1/i) \le x < 5/2$.

Then $\{A_i\}$, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 1$.

As a corollary of above theorem we get the following:

Corollary. Let A_1 , A_2 , S and T be selfmappings of a complete metric space (X,d) such that for $\alpha , \beta > 0$ with $\alpha + \beta < 1$ and for all x , y in X , we have

(i) $A_1X \subseteq TX$ and $A_2X \subseteq SX$

(ii) $$d(A_1x,A_2y) \le \frac{\alpha\, \phi(d(A_2y,Ty))\,[1 + \phi(d(A_1x,Sx))]}{[1 + \phi(d(Sx,Ty))]} + \beta\, \phi(d(Sx,Ty))$$

where $\phi : R_+ \to R_+$ denote an upper semicontinuous function such that $\phi(t) < t$ for each $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$. Let (A_1,S) and (A_2,T) be compatible pairs . If the mappings in one of the compatible pairs are reciprocally continuous then A_1 , A_2 , S and T have a unique common fixed point.

The proof follows directly from the proof of the above theorem and requires no additional argument.

REFERENCES

[1] Fisher B., *Common fixed point of four mappings,* Bull. Inst. Math. Acad. Sinica 11(1983), 103-113.
[2] Jachymski J., *Common fixed point theorems for some families of maps,* Indian J. Pure Appl. Math. , 25 (1994), 925-937.
[3] Jungck G., *Compatible mappings and common fixed points,* Internat. J. Math. and Math. Sci. , 9 (1986), 771-779.
[4] Jungck G., Moon K.B., Park S. and Rhoades B.E., *On generalizations of the Meir-Keeler type contraction maps,* J. Math. Anal. Appl. , 180 (1993) , 221-222 .
[5] Pant R.P., *Common fixed points of two pairs of commuting mappings,* Indian J. Pure Appl. Math. , 17 (1986), 187-192.
[6] Pant R.P., *Common fixed points of weakly commuting mappings,* Math. Student, 62 (1993), 97-102.
[7] Pant R.P., *Common fixed points of sequences of mappings,* Ganita, 47 (1996), 43-49 .
[8] Rao I.H.N. and Rao K.P.R., *Generalizations of the fixed point theorem of Meir and keeler type,* Indian J. Pure Appl. Math. , 16 (1985), 1249-1262.
[9] Rhoades B.E., Park S. and Moon K.B., *On generalizations of the Meir-Keeler type contraction maps* J. Math. Anal. Appl. , 146 (1990), 482-494.
[10] Pant R.P., *A common fixed point theorem under a new condition,* Indian J. Pure Appl. Math. , 30(2) (1999) 147-152.