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Oscillation of Third-Order Nonlinear Neutral Delay Dynamic Equations on Time Scales

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Abstract:

In this paper, we establish some sufficient conditions for oscillation of solution of third order nonlinear neutral delay dynamic equations of the type

$$(a(t)((x(t)+p(t)x(\delta(t)))^{\Delta^2})^{\gamma})^{\Delta}+q(t)x^{\gamma}(\tau(t))=0.$$

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I. INTRODUCTION

We consider the following nonlinear neutral delay dynamic equations of the form

$$(a(t)((x(t)+p(t)x(\delta(t)))^{\Delta^2})^{\gamma})^{\Delta}+q(t)x^{\gamma}(\tau(t))=0. \qquad (1)$$

Throughout this paper we assume the following conditions:

(H) γ is a ratio of odd positive integers. $a(t), p(t), q(t)$ are positive real-valued rd-continuous functions defined on the time scale interval $[a,b]$ (throughout $a,b\in \mathrm{T}$ with $a<b$), $0\le p(t)\le p<1$, $\tau(t)\le t$ and $\delta(t)\le t$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$ are positive and $\int_{t_0}^{\infty}\frac{1}{a^{\frac{1}{\gamma}}(s)}\Delta s=\infty$ and set $z(t)=x(t)+p(t)x(\delta(t))$.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D thesis [7] in order to unify continuous and discrete analysis. A time scale î is an arbitrary nonempty closed subset of the reals, and scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models (see [3]). A book on the subject of time scales by Bohner and Peterson [3] summarizes and organizes much of the time scale calculus.

Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a,\infty)$. By a solution of (1) we mean a nontrivial real-valued of function x satisfying equation (1) for $t\ge a$ solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called non oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1) which exist on some half line $[t_x,\infty)$ and satisfy $\sup\{|x(t)|:t>t_0\}>0$ for any $t_0>t_x$.

In this paper, we obtain some oscillation criteria for (1). This paper is organized as follows. In the next section, we present some basic definitions concerning the calculus on time scales. In section 3, we will use the Riccati transformation technique to give some sufficient conditions which guarantee that every solution of (1) is oscillatory or converges to zero.

II. SOME PRELMINARIES ON TIME SCALES AND SOME LEMMAS

In this section, we present some basic definitions concerning the calculus on time scales which are contained in [3], and then we state and prove some lemmas which we will need in the proofs of our main results.

A time scale î is an arbitrary nonempty closed subset of the real numbers . On any time scale î we define the forward jump operator σ and the graininess function μ by

$$\sigma(t)=\inf\{s\in T : s>t\} \text{ and } \mu(t)=\sigma(t)-1.$$

A point $t\in T$ with $\sigma(t)=t$ is called right-dense, while t is referred to as being right-scattered if $\sigma(t)>t$. The backward jump operator and left-dense and left-scattered points are defined in a similar way.

A function $f:T\to$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points. The (Δ) derivative of $f:T\to T$ at a right-dense point t is defined by.

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

Provided this limit exist, and if t is right-scattered and *f* is continuous at *t*, we define the (Δ) derivative at *t* by

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$$

The derivative and the shift operator σ are related by the useful formula

$$f^{\sigma}=f+\mu f^{\Delta}, \text{ where } f^{\sigma}(t)=f(\sigma(t)). \qquad (2)$$

We will make use of the following product and quotient rules for the derivative of the produce fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma}\neq 0$) of two differentiable function *f* and *g*:

$$(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta} \text{ and } \left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\Delta}}. \qquad (3)$$

By using the product rule form (3), the derivative of $f(t)=(t-\alpha)^{m}$ for m ∈ ℕ, and α ∈ R can be calculated (see [[3], Theorem 1.24]) as

$$f^{\Delta}(t)=\sum_{v=0}^{m-1}(\sigma(t)-\alpha)^{v}(t-\alpha)^{m-v-1}. \qquad (4)$$

For a, b ∈ î, and a differentiable function *f*, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t=f(b)-f(a).$$

The integration by parts formula follows from (3)

$$\int_a^b f(t)g^{\Delta}(t)\Delta t=[f(t)g(t)]_a^b-\int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t. \qquad (5)$$

Now, we state and prove some useful lemmas, which we will use in the proofs of our main results. We begin with the following lemma.

III. MAIN RESULTS

Lemma 3.1. *Let $x(t)$ be a positive solution of (1). Then there are only the following two cases for* $z(t)=x(t)+p(t)x(\delta(t))$

(i) $z(t)>0, z^{\Delta}(t)>0, z^{\Delta^2}(t)>0,$

(ii) $z(t)>0, z^{\Delta}(t)<0, z^{\Delta^2}(t)>0, t\in[t_1,\infty)$ *where t_1 is sufficiently large.*

Proof. Assume that $x(t)$ is a positive solution of (1) on $t\in[t_0,\infty).$ We see that $z(t)>x(t)>0$ and

$$\left(a(t)\left(\left(z(t)\right)^{\Delta^2}\right)^{\gamma}\right)^{\Delta}=-q(t)x^{\gamma}(\tau(t))<0 \qquad (6)$$

$a(t)\left(\left(z(t)\right)^{\Delta^2}\right)^{\gamma}$ is decreasing and of one sign. Therefore $z^{\Delta^2}(t)$ is also of one sign. We have two possibilities; $z^{\Delta^2}(t)<0$ or $z^{\Delta^2}(t)>0$ for $t\in[t_1,\infty)$ by (6). If we choose $z^{\Delta^2}(t)<0,$ then there exists a constant $M>0$ such that

$$a(t)\left(\left(z(t)\right)^{\Delta^2}\right)^{\gamma}\leq -M<0$$

Integrating the above inequality from t_1 to t, we obtain

$$z^{\Delta}(t)\leq z^{\Delta}(t_1)-M^{\frac{1}{\gamma}}\int_{t_1}^{t}\frac{1}{a^{\frac{1}{\gamma}}(s)}\Delta s.$$

Letting $t\to\infty$ and using (H), we obtain $z^{\Delta}(t)\to-\infty$. Thus $z^{\Delta}(t)<0$ eventually. But $z^{\Delta^2}(t)<0$ and $z^{\Delta}(t)<0$ eventually. Hence $z(t)<0$ for $t\in[t_1,\infty),$ which is a contradiction. This contradiction proves $z^{\Delta^2}(t)>0$ and we have only two cases, (i), (ii) for $z(t)$. The proof is complete.

Lemma 3.2. *Let $x(t)$ be a positive solution of equation (1) and the corresponding $z(t)$ satisfy (ii). If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(s)\Delta s\right)^{\frac{1}{\gamma}}\Delta u\Delta v=\infty, \tag{7}$$

Then $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}z(t)=0.$

Proof. Suppose that $x(t)$ is a positive solution of (1). Since $z(t)>0$ and $z^{\Delta}(t)<0$, then there exists a finite limit, $\lim_{t\to\infty}z(t)=\ell$. We shall prove that $\ell=0$. Assume that $\ell>0$. Then for any $\varepsilon>0$, we have $\ell+\varepsilon>z(t)>\ell$ eventually. Choose $0<\varepsilon<\dfrac{\ell(1-p)}{p}$. It is easy to verify that

$$x(t)=z(t)-p(t)x(\delta(t))>\ell-p(\ell+\varepsilon)=k(\ell+\varepsilon)>kz(t)$$

where $k=\dfrac{\ell-p(\ell+\varepsilon)}{\ell+\varepsilon}>0$. Using the above inequality, we obtain from (6)

$$\left(a(t)\left(\left(z(t)\right)^{\Delta^2}\right)^{\gamma}\right)^{\Delta}\le -q(t)k^{\gamma}z^{\gamma}(\tau(t)).$$

Integrating the above inequality from t to ∞, we obtain

$$a(t)\left(\left(z(t)\right)^{\Delta^2}\right)^{\gamma}\ge k^{\gamma}\int_{t}^{\infty}q(s)z^{\gamma}(\tau(s))\Delta s.$$

Using $z^{\gamma}(\tau(t))\ge \ell^{\gamma}$, we get $z^{\Delta^2}(t)\ge k\ell\left(\dfrac{1}{a(t)}\displaystyle\int_{t}^{\infty}q(s)\Delta s\right)^{\frac{1}{\gamma}}$.

Integrating again from t to ∞, we have $-z^{\Delta}(t)\ge k\ell\displaystyle\int_{t}^{\infty}\left(\dfrac{1}{a(u)}\int_{u}^{\infty}q(s)\Delta s\right)^{\frac{1}{\gamma}}\Delta u.$

Again integrating from t_1 to ∞, we obtain $z(t_1)\ge k\ell\displaystyle\int_{t_1}^{\infty}\int_{u}^{\infty}\left(\dfrac{1}{a(u)}\int_{v}^{\infty}q(s)\Delta s\right)^{\frac{1}{\gamma}}\Delta u\Delta v.$

This contradicts (7). Then $\ell=0$. Moreover the inequality $0\le x(t)\le z(t)$ implies that $\lim_{t\to\infty}x(t)=0$ and the proof is complete.

Lemma 3.3. *Assume that $u(t)>0, u^{\Delta}(t)\ge 0, u^{\Delta^2}(t)\le 0$ for all $t\in[t_0,\infty)$. Then for each $\ell\in(0,1)$ there exists an integer $T\ge t_0$ such that $\dfrac{u(\tau(t))}{\tau(t)}\ge \ell\dfrac{u(t)}{t}$ for $t\ge T$.*

Proof. From the monotonicity property of $u^{\Delta}(t)$, we have

$$u(t)-u(\tau(t))=\int_{\tau(t)}^{t}u^{\Delta}(s)\Delta s\le u^{\Delta}(\tau(t))(t-\tau(t))$$

$$\frac{u(t)}{u(\tau(t))}\le 1+\frac{u^{\Delta}(\tau(t))}{u(\tau(t))}(t-\tau(t)) \tag{8}$$

Also

$$u(\tau(t))\ge u(\tau(t))-u(t_0)\ge (u^{\Delta}(\tau(t)))(\tau(t)-t_0).$$

So, for each $\ell\in(0,1)$ and $T\ge t_0$ such that

$$\frac{u(\tau(t))}{u^{\Delta}(\tau(t))}\ge \ell\,\tau(t), t\ge T. \tag{9}$$

Combining (8) and (9), we get

$$\frac{u(t)}{u(\tau(t))}\le 1+\frac{1}{\ell(\tau(t))}(t-\tau(t)).$$

$$\le\left(1-\frac{1}{\ell}\right)+\frac{t}{\ell(\tau(t))}$$

$$\frac{u(\tau(t))}{u^{\Delta}(\tau(t))}\le\frac{t}{\ell(\tau(t))}$$

and the proof is complete.

Lemma 3.4. *Assume that* $z(t)>0, z^{\Delta}(t)>0, z^{\Delta^2}(t)>0, z^{\Delta^3}(t)\le 0$ *for all* $t\ge T$*. Then* $\frac{z(t)}{z^{\Delta}(t)}\ge\frac{t-T}{2}$ *for* $t\ge T$.

Proof. From the monotonicity property of $z^{\Delta^2}(t)$, we have

$$z^{\Delta}(t)=z^{\Delta}(T)+\int_T^t z^{\Delta^2}(s)\Delta s\ge(t-T)z^{\Delta^2}(t).$$

Integrating from T to t, we obtain

$$z(t)-z(T)\ge\int_T^t (s-T)z^{\Delta^2}(s)\Delta s$$

$$z(t)\ge z(T)+(t-T)z^{\Delta}(t)-z(t)+z(T)$$

$$z(t)\ge\frac{1}{2}(t-T)z^{\Delta}(t)$$

Lemma 3.5. *Assume that* $z(t)>0, z^{\Delta}(t)>0, z^{\Delta^2}(t)>0, z^{\Delta^3}(t)\le 0$ *for all* $t\ge T$. *Then* $(t-T)\frac{z^{\Delta^2}(t)}{z^{\Delta}(t)}\le 1$ *for* $t\ge T$.

Proof. The result follows from the following inequality

$$z^{\Delta}(t)\ge\int_T^t z^{\Delta^2}(s)\Delta s\ge z^{\Delta^2}(t)(t-T).$$

Now, we present the oscillation results. For simplicity, we introduce the following notation

$$p_*=\liminf_{t\to\infty}\frac{t^{\gamma}}{a(t)}\int_t^{\infty}P_{\ell}(s)\Delta s,\quad q_*=\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\frac{s^{\gamma+1}}{a(s)}P_{\ell}(s)\Delta s, \tag{10}$$

where $P_{\ell}(s)=\ell^{\gamma}(1-p)^{\gamma}q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}\left(\frac{\tau(s)-T}{2}\right)^{\gamma}$ with $\ell\in(0,1)$ arbitrarily chosen and T large enough.

Moreover for $z(t)$ satisfying case (i), we define

$$w(t)=a(t)\left(\frac{z^{\Delta^2}(t)}{z^{\Delta}(t)}\right)^{\gamma} \tag{11}$$

and

$$r=\liminf_{t\to\infty}\frac{t^{\gamma}w^{\sigma}(t)}{a^{\sigma}(t)}\text{ and }R=\limsup_{t\to\infty}\frac{t^{\gamma}w(t)}{a(t)}. \tag{12}$$

Lemma 3.6. *Assume that* $a(t)$ *is non- decreasing. Let* $x(t)$ *be a positive solution of equation (1)*

(I) Let $p_*<\infty$ *and* $q_*<\infty$*. Suppose that the corresponding* $z(t)$ *satisfies case (i) of Lemma 3.1. Then*

$$p_*\le r-r^{1+\frac{1}{\gamma}}\text{ and }p_*+q_*\le 1 \tag{13}$$

(II) If $p_*=\infty$ *or* $q_*=\infty$, *then* $z(t)$ *does not belong under case (i) of Lemma 3.1.*

Proof. Part (I): Assume that $x(t)$ is a positive solution of equation (1) and the corresponding $z(t)$ satisfies (i). First note that

$$x(t)=z(t)-p(t)x(\delta(t))>z(t)-p(t)z(\delta(t))\ge(1-p)z(t).$$

Using the above inequality in equation (1), we obtain

$$\left(a(t)((z(t))^{\Delta^2})^{\gamma}\right)^{\Delta}\le-(1-p)^{\gamma}q(t)z^{\gamma}(\tau(t))\le 0 \tag{14}$$

The last inequality together $a^{\Delta}(t) \geq 0$ gives $z^{\Delta^3}(t) \leq 0$. So there exists a $T \geq t_0$ such that $z(t)$ satisfies $z(\tau(t)) > 0, z^{\Delta}(t) > 0, z^{\Delta^2} > 0, z^{\Delta^3}(t) \leq 0$, for $t \geq T.$ From definition of $w(t)$ and (14) we see that $w(t)$ is positive and satisfies,

$$w^{\Delta}(t) = \frac{\left(a(t)(z^{\Delta^2}(t))^{\gamma}\right)^{\Delta}}{\left(z^{\Delta}(t)\right)^{\gamma}} - \frac{a^{\sigma}(t)\left(z^{\Delta^2\sigma}(t)\right)^{\gamma}\left(\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}}{\left(z^{\Delta}(t)\right)^{\gamma}\left(z^{\Delta\sigma}(t)\right)^{\gamma}}$$

$$w^{\Delta}(t) \leq \frac{-(1-p)^{\gamma} q(t) z^{\gamma}(\tau(t))}{(z^{\Delta}(t))^{\gamma}} - \frac{\gamma}{a^{\frac{1}{\gamma}\sigma}(t)} w^{\frac{\gamma+1}{\gamma}\sigma}(t) \qquad (15)$$

From Lemma 3.3 with $u(t) = z^{\Delta}(t),$ we have for ℓ the same as in P_{ℓ}

$$\frac{1}{z^{\Delta}(t)} \geq \frac{\ell\tau(t)}{t}\frac{1}{z^{\Delta}(\tau(t))}, t \geq T,$$

which with (15) gives $w^{\Delta}(t) \leq -\ell^{\gamma} q(t)\left(\frac{\tau(t)}{t}\right)^{\gamma}\left(\frac{z(\tau(t))}{z^{\Delta}(\tau(t))}\right)^{\gamma}(1-p)^{\gamma} - \frac{\gamma}{a^{\frac{1}{\gamma}\sigma}(t)} w^{\frac{\gamma+1}{\gamma}\sigma}(t)$

Using the fact form Lemma 3.4 that $z(t) \geq \frac{(t-T)}{2} z^{\Delta}(t)$ we have,

$$w^{\Delta}(t) + P_{\ell}(t) + \frac{\gamma}{a^{\frac{1}{\gamma}\sigma}(t)} w^{\frac{\gamma+1}{\gamma}\sigma}(t) \leq 0. \qquad (16)$$

Since $P_{\ell}(t) > 0$ and $w(t) > 0$ for $t \geq T, w^{\Delta}(t) \leq 0$ and $\frac{-w^{\Delta}(t)}{\gamma w^{\frac{\gamma+1}{\gamma}\sigma}(t)} \geq \frac{1}{a^{\frac{1}{\gamma}\sigma}(t)}$ for $t \geq T$.

This implies that $\left(\frac{1}{w^{\frac{1}{\gamma}}(t)}\right)^{\Delta} \geq \frac{1}{a^{\frac{1}{\gamma}\sigma}(t)}.$

Integrating the last inequality T to t and using the fact that $w(t)$ is decreasing, we obtain

$$w(t) < \frac{1}{\left(\gamma \int_T^t \frac{1}{a^{\frac{1}{\gamma}\sigma}(s)} \Delta s\right)^{\gamma}} \qquad (17)$$

which view of (H) implies that, $\lim_{t\to\infty} w(t) = 0.$ On the other hand, from the definition of $w(t)$ and Lemma 3.5, we see that

$$0 \leq r \leq R \leq 1. \qquad (18)$$

Now, we prove that the first inequality in (13) holds. Let $\in > 0$, then from the definition of p_* and r, we can pick up $t_2 \in [T, \infty)$ sufficiently large $\frac{t^{\gamma}}{a(t)}\int_t^{\infty} P_{\ell}(s)\Delta s \geq p_* - \varepsilon$ and $\frac{t^{\gamma} w^{\sigma}(t)}{a^{\sigma}(t)} \geq r - \varepsilon$ for $t \in [t_2, \infty).$ Integrating (16) from t to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w(t) \geq \int_t^{\infty} P_{\ell}(s)\Delta s + \gamma\int_t^{\infty} \frac{w^{\frac{\gamma+1}{\gamma}\sigma}(s)}{a^{\frac{1}{\gamma}\sigma}(s)}\Delta s, \quad for\ t \geq t_2 \qquad (19)$$

Using the fact $a^{\Delta}(t) \geq 0,$ it follows from (19) that

$$\frac{t^{\gamma} w(t)}{a(t)} \geq (p_* - \varepsilon) + \frac{\gamma t^{\gamma}}{a(t)}\int_t^{\infty} \frac{s^{\gamma+1} a^{\sigma}(s) w^{\frac{\gamma+1}{\gamma}\sigma}(s)}{s^{\gamma+1} a^{\frac{\gamma+1}{\gamma}\sigma}(s)}\Delta s$$

and so

$$\frac{t^{\gamma} w(t)}{a(t)} \geq (p_* - \varepsilon) + t^{\gamma} (r-\varepsilon)^{1+\frac{1}{\gamma}} \int_t^{\infty} \frac{\gamma}{s^{\gamma+1}} \Delta s. \qquad (20)$$

From (20), we have $\frac{t^{\gamma} w(t)}{a(t)} \geq (p_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\gamma}}$

Taking lim inf of both sides as $t \to \infty$, we get $r \geq (p_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\gamma}}$

Since, $\in > 0$ is arbitrary we get the result,

$$p_* \leq r - r^{1+\frac{1}{\gamma}} \qquad (21)$$

To complete the proof of Part (I), it remains to prove second inequality in (13). Multiplying the inequality (16) by $\frac{t^{\gamma+1}}{a(t)}$ and integrating from t_2 to t, we obtain

$$\int_{t_2}^{t} \frac{s^{\gamma+1} w^{\Delta}(s)}{a(s)} \Delta s \leq -\int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} P_{\ell}(s) \Delta s - \gamma \int_{t_2}^{t} \left(\frac{s^{\gamma} w^{\sigma}(s)}{a^{\sigma}(s)} \right)^{\frac{\gamma+1}{\gamma}} \Delta s. \qquad (22)$$

By integration by parts, we obtain

$$\frac{t^{\gamma+1}}{a(t)} w(t) \leq \frac{t_2^{\gamma+1}}{a(t_2)} w(t_2) - \int_{t}^{t} \frac{s^{\gamma+1}}{a(s)} P_{\ell}(s) \Delta s - \gamma \int_{t_2}^{t} \left(\frac{s^{\gamma} w^{\sigma}(s)}{a^{\sigma}(s)} \right)^{\frac{\gamma+1}{\gamma}} \Delta s + \int_{t_2}^{t} w^{\sigma}(s) \left(\frac{s^{\gamma+1}}{a(s)} \right)^{\Delta} \Delta s.$$

Since $a^{\Delta}(t) \geq 0,$ we have

$$\left(\frac{s^{\gamma+1}}{a(s)} \right)^{\Delta} = \frac{a(s)(s^{\gamma+1})^{\Delta} - s^{\gamma+1} a^{\Delta}(s)}{a(s) a^{\sigma}(s)} \leq \frac{(\gamma+1)\sigma^{\gamma}(s)}{a^{\sigma}(s)}$$

Hence,

$$\frac{t^{\gamma+1}}{a(t)} w(t) \leq \frac{t_2^{\gamma+1}}{a(t_2)} w(t_2) - \int_{t_2}^{t} \frac{s^{\gamma+1} P_{\ell}(s)}{a(s)} \Delta s + \int_{t_2}^{t} \left[\frac{(\gamma+1)\sigma(s)^{\gamma} w^{\sigma}(s)}{a^{\sigma}(s)} - \gamma \left(\frac{s^{\gamma} w^{\sigma}(s)}{a^{\sigma}(s)} \right)^{\frac{\gamma+1}{\gamma}} \right] \Delta s \,.$$

Using the inequality $Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}$ with $u = \frac{s^{\gamma} w^{\sigma}(s)}{a^{\sigma}(s)} > 0, A = \gamma$ and $B = (\gamma+1)\left(\frac{\sigma(s)}{s} \right)^{\gamma}$, we get

$$\frac{t^{\gamma+1}}{a(t)} w(t) \leq \frac{t_2^{\gamma+1}}{a(t_2)} w(t_2) - \int_{t_2}^{t} \frac{s^{\gamma+1} P_{\ell}(s)}{a(s)} \Delta s + \int_{t_2}^{t} \left(\frac{\sigma(s)}{s} \right)^{\gamma(\gamma+1)} \Delta s$$

It follows that

$$\frac{t^{\gamma} w(t)}{a(t)} \leq \frac{t_2^{\gamma+1} w(t_2)}{t a(t_2)} - \frac{1}{t} \int_{t_2}^{t} \frac{s^{\gamma+1} P_{\ell}(s)}{a(s)} \Delta s + \frac{1}{t} \int_{t_2}^{t} \left(\frac{\sigma(s)}{s} \right)^{\gamma(\gamma+1)} \Delta s. \qquad (23)$$

Taking the lim sup of $t \to \infty$

$$R \leq -q_* + 1.$$

Combining this with the inequalities in (21) and (18), we have

$$p_* \leq r - r^{1+\frac{1}{\gamma}} \leq r \leq R \leq -q_* + 1$$

which gives the desired second inequality in (13). The proof of part (I) is complete.

Part (II): Assume $x(t)$ positive solution of (1). We shall show that $z(t)$ does not belong to case (i) of Lemma 3.1. Assume the contrary. First assume $p_* = \infty$. This is exactly as in the proof of the first part, we obtain (19). Then

$$\frac{t^{\gamma}}{a(t)} w(t) \geq \frac{t^{\gamma}}{a(t)} \int_t^{\infty} P_{\ell}(s) \Delta s.$$

Taking lim inf of both sides at $t \to \infty$, we obtain in view of (18)

$$1 \ge r \ge \infty.$$

This is a contradiction. Next we assume that $q_* = \infty$. Then taking lim inf and lim sup on the left and right sides of (23) respectively, we obtain

$$0 \le R \le -\infty..$$

This contradiction completes the proof.
Now we are ready to present the following oscillation criterion for equation (1)
Theorem 3.7. *Assume that condition (7) holds and a(t) is non-decreasing. Let* $x(t)$ *be a solution of (1). If*

$$p_* = \liminf_{t\to\infty} \frac{t^\gamma}{a(t)} \int_t^\infty P_\ell(s)\Delta s > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \qquad (24)$$

Then $x(t)$ *is oscillatory or* $x(t) \to 0$ *as* $t \to \infty$.
Proof. Let $x(t)$ be a non-oscillatory solution of equation (1). Without loss of generality we may assume that $x(t)$ is a positive solution of equation (1). If $p_* = \infty$, then by Lemma 3.6, $z(t)$ does not belong to case (i) of Lemma 3.1. we see that $\liminf_{t\to\infty} x(t) = 0$.

Next, we assume that $p_* < \infty$. We shall discuss two possibilities. If for $z(t)$ case (ii) holds, then exactly as above we are led, by Lemma 3.1, to $\liminf_{t\to\infty} x(t) = 0$.

Now we assume that for $z(t)$ case (i) holds. Let $w(t)$ and r be defined by (11) and (12) respectively, then from Lemma 3.6 we see that r satisfies the inequality

$$p_* \le r - r^{1+1/\gamma}.$$

Using the inequality $Bu - Au^{1+1/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} B^{\gamma+1} / A^\gamma$ with A = B = 1 and $u = r$, we obtain that

$$p_* \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

which contradicts (24). This completes the proof.
Corollary 3.8. *Assume that condition (7) holds and* $a(t)$ *is non-decreasing. Let* $x(t)$ *be a solution of (1). If*

$$\liminf_{t\to\infty} \frac{t^\gamma}{a(t)} \int_t^\infty q(s) \frac{\tau^{2\gamma}(s)}{s^\gamma} \Delta s > \frac{(2\gamma)^\gamma}{(\gamma+1)^{\gamma+1}(1-p)^\gamma}, \qquad (25)$$

then $x(t)$ *is oscillatory or* $x(t) \to 0$ *as* $t \to \infty$.
Proof. We shall show that condition (25) implies condition (24). First note that for any $\ell \in (0,1)$ there exists an integer t_1 such that $\tau(t) - T \ge \ell\,\tau(t), t \ge t_1$. Therefore,

$$P_\ell(t) \ge \frac{\ell^{2\gamma}(1-p)^\gamma}{2^\gamma} \frac{\tau^{2\gamma}(t)}{t^\gamma} q(t), t \ge t_1. \qquad (26)$$

On the other hand, (25) implies that for some $\ell \in (0,1)$

$$\liminf_{t\to\infty} \frac{t^\gamma}{a(t)} \int_t^\infty q(s) \frac{\tau^{2\gamma}(s)}{s^\gamma} \Delta s > \frac{1}{\ell^{2\gamma}} \frac{(2\gamma)^\gamma}{(\gamma+1)^{\gamma+1}(1-p)^\gamma}, \qquad (27)$$

Combining (26) with (27), we obtain (24).
Theorem 3.9. *Assume that condition (7) holds and* $a(t)$ *is non-decreasing. Let* $x(t)$ *be a solution of equation (1). If*

$$p_* + q_* > 1, \qquad (28)$$

then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a non oscillatory solution of equation (1). Without loss of generality we may assume that $x(t)$ is a positive solution of equation (1). If $p_* = \infty$ or $q_* = \infty$, then by Lemma 3.6, $z(t)$ does not belong to case (i) of Lemma 3.1. That is, $z(t)$ has to satisfy case (ii). From Lemma 3.2, we see that $\liminf_{t\to\infty} x(t) = 0$.

Next, we assume that $p_* < \infty$ or $q_* < \infty$. We shall discuss two possibilities. If for $z(t)$ case(ii) holds, then exactly as above we are led, by Lemma 3.2, to $\liminf_{t\to\infty} x(t) = 0$. Now we assume that for $z(t)$ case (i) holds. Let $w(t)$ and r be defined by (11) and (12), respectively. Then from Lemma 3.6 we see that p_* or q_* satisfy the inequality $p_* + q_* \le 1,$ which contradicts (28). This completes the proof.

As a consequence of Theorem 3.9, we have the following results.

Corollary 3.10. *Assume that condition (7) holds and* $a(t)$ *is non decreasing. Let* $x(t)$ *be a solution of equation (1). If*

$$q_* = \lim_{t\to\infty}\sup \frac{1}{t}\int_{t_0}^{t} \frac{s^{\gamma+1}}{a(s)} P_\ell(s)\Delta s > 1, \qquad (29)$$

Then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0.$

As a matter of fact we can again slightly simplify function $P_\ell(t)$ in (25).

Corollary 3.11. *Assume that condition (7) holds and* $a(t)$ *is non decreasing. Let* $x(t)$ *be a solution of equation (1). If*

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_{t_0}^{t} \frac{s\tau^{2\gamma}(s)}{a(s)} q(s)\Delta s \ge \frac{2^\gamma}{(1-p)^\gamma} \qquad (30)$$

Then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0.$

The proof is similar to that of Corollary 3.8 and hence the details are omitted.

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