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Spectral Theorem for Compact Self Adjoint Operators

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Abstract:

Spectral Theorem provides spectral decomposition, Eigen value decomposition of the underlying vector space on which the operator acts. Here, we have tried to work on the formulation of an operator explicitly, operator being self adjoint and compact defined on Hilbert space.

Keywords: UDU, Hilbert Space H

I. INTRODUCTION

Spectral Theory for a self adjoint operator is quite complicated. But it becomes easier if the operator at hand is compact. Consider on operator. $T : H \rightarrow H$ with H being a Hilbert space. The complete spectral decomposition of T can be stated in a quite elementary fashion. Spectral Theorem is a generalization of the familiar theorem from Linear algebra asserting that a self adjoint n x n matrix A can be diagnolized Ithat is there is a diagonal matrix D and unitary matrix U st. $A = UDU^{-1}$,In particular a compact self adjoint operator can be unitarily diagonalized. Actually, spectral theory is an inclusive term for theories extending the Eigen vector and Eigen value theory of single matrix to a much broader theory of operators in a variety of mathematical spaces. This project is concerned with studying the "spectral representation of compact self adjoint operators. Here, we started with proving the existence of unit vector x_0 of H with $\|Tx_o\| = \|T\|$. Then we proved the fact that T has an Eigen value $\|T\|$ or $-\|T\|$, where T is a compact and self adjoint operator defined on Hilbert space H. Then we found the representation of T in the main proof.

Important Definitions

1. **Compact Operator:** A Linear operator $A: X \rightarrow Y$ is said to be compact if the set $Cl\{ Ax : \|x\| \leq 1\}$ is compact in Y.
2. **Self Adjoint Operator:** A bounded operator A on a Hilbert space H is said to be self adjoint if $A^* = A$, where A^* is adjoint of A.
3. **Kernel of an Operator:** If $A : X \rightarrow Y$ b a Linear operator then $N(A) = \{ x \in X : Ax = 0\}$ is called kernel of A or null space of A.
4. **Cauchy's Sequence:** A sequence $< x_n >$ in X is said to be a cauchy's sequence if for every $\in > 0$ $\exists$ $p \in N$ such that
5. **Projection:** A linear operator $P : X \rightarrow X$ is called a projection operator or simple projection if $Px = x$ $\forall$ $x \in R(P)$

II. SPECTRAL THEOREM

We will discuss the spectral theorem of compact self adjoint operators. Firstly, we will give statement of the theorem. For proving the theorem, we require some additional results which will be discussed in the subsequent sections.

Statement of Theorem:

Let $T : H \rightarrow H$ be a compact and self adjoint operator on a Hilbert space H.
Then there is a finite or infinite sequence $\{\lambda_n\}_{n=1}^{N}$ ($n \in Z^+$ Or $N = \infty$) of real eigen values $\lambda_n \neq 0$ and a corresponding orthonormal sequence $\{e_n\}_{n=1}^{N}$ in H such that

(a) $Te_n = \lambda_n e_n$ $\forall$ n with $1 \leq n \leq N$
(b) $N(T) = span(\{e_n\}_{n=1}^{N}$
(c) if $N = \infty$ then $\lambda_n = 0$ as $n \rightarrow \infty$

Before going further, we will prove some very important results for the theorem which are given in the form of lemma:

Lemma 1: If X be a reflexive normed space and X is separable. Then every bounded sequence $<x_n>$ in X has a subsequence which is weakly convergent.

Lemma 2 :Let $T: H \rightarrow H$ be a compact and self adjoint operator on a Hilbert space H. Let. $S_1 = S(0,1)$ be unit sphere in H. Then there is a vector $x_0 \in S_1$ such that $\|Tx_0\| = \|T\|$

Lemma 3: If $T : H \rightarrow H$ be compact and self adjoint operator on Hilbert Space H. Then T has an eigen vector with eigen value $\|T\|$ or $-\|T\|$

Lemma 4: Let $T : H \rightarrow H$ be a bounded self adjoint operator on a Hilbert space H. and Let $Y \subset H$, be a subspace s.t $T(Y) \subset Y$. Then $T(Y^{\perp}) \subset Y^{\perp}$ and $T|_{Y^{\perp}} : Y^{\perp} \rightarrow Y^{\perp}$ is a bounded self adjont operator on Hilbert space $Y^{\perp}$ with norm $\|Ty^{\perp}\| \leq \|T\|$

Lemma 5: Let X be a finite dimensional inner product space and T is a self adjoint operator on X, Then T can be represented as T = $\int_A^B \lambda \, dE_\lambda$

First Proof of the Theorem:

In this section , we start with finding eigen vectors and then restrict the attention to the orthogonal complement of this set of vectors and then prove the required result as given here. We have taken the analogy between the spectral theory of operators on Hilbert spaces and that of operators on finite dimensional spaces about as far as it will go without requiring serious modifications. Firstly, Let us suppose T = 0, then the theorem is trivial. So, suppose T ≠ 0, Then by Lemma 3, there is an Eigen vector e_1 with Eigen value such that $\lambda_1 = \| T \|$ or $\lambda_1 = - \| T \|$. Since T ≠ 0, we have $\lambda_1 \neq 0$

Let us assume $\| e_1 \| = 1$ Let $H_1 = \text{span}\{e_1\}^{\perp}$ Then by Lemma 4, the restriction map $T_{|H1}$ is a self adjoint operator on H_1 and $\| T_{|H1|} \| \leq \| T \| = \lambda$ If $T_{|H1} = 0$, then we stop here. If not, we repeat the above process. After n-1 steps, we found an orthonormal sequence of Eigen vectors $e_1, e_2, \ldots\ldots e_{n-1}$ in H with corresponding real Eigen values $\lambda_1, \lambda_2 \ldots. \lambda_{n-1}$ s .t.

$|\lambda_1| \geq |\lambda_2| \geq \ldots\ldots\ldots\ldots \geq |\lambda_{n-1}| > 0$ Let $H_{n-1} = \text{span}\{e_1, e_2 \ldots. e_{n-1}\}^{\perp}$ then $T_{|H\,n-1}$ is a self adjoint operator defined on H_{n-1} and $\| T_{IH\,n-1} \| \leq |\lambda_{n-1}|$

Now if $T_{|Hn-1} = 0$, then we stop after this step. and if $T_{|Hn-1} \neq 0$, we continue to nth step and again apply Lemma 3, we find an Eigen vector $e_n \in H_{n-1}$ with eigen value $\lambda_n = \| T_{|Hn-1} \|$ or $\lambda_n = - \| T_{|Hn-1} \|$. Here, we have $|\lambda_n| \leq |\lambda_{n-1}|$. Let us assume $\| e_n \| = 1$

Because $e_n \in H_{n-1} = \text{span}\{ e_1, e_2 \ldots e_{n-1} \}^{\perp}$ We have $< e_n, e_k > = 0$ for k = 1, 2, .. n -1 i.e. $e_1, e_2, \ldots e_n$ is an orthonormal sequence. Let $H_n = \text{span}\{e_1, e_2 \ldots e_n\}^{\perp}$. Then again by lemma 4, the restriction map $T_{|Hn}$ is self adjoint operator on H_n since, $H_n \subset H_{n-1}$, We have $\| T_{|Hn} \| \leq \| T_{|.Hn-1} \| = |\lambda_n|$. which is the same situation as above.

Now, if the process stops after step N, we have $T_{|HN} = 0$ This implies, span $(\{e_n\}^N_{n=1}) = H_N \subset N(T)$ …………….(a)

Again if $x \in N(T) \Rightarrow Tx = 0$

for Each n, $\lambda_n <e_n, x> = < \lambda_n e_n, x> = <Te_n, x> = <e_n, Tx> = <e_n, o> = 0$

i.e. $\lambda_n < e_n, x> = 0 \Rightarrow x \perp e_n \Rightarrow x \in \text{span}(\{e_n\}^N_{n=1})^{\perp}$ ……………………………(b)

combining (a) and (b), $N(T) = \text{span}(\{e_n\}^N_{n=1})^{\perp}$ which is second part of the theorem.

Now, if the process never steps, we obtain an infinite sequence $\{\lambda_n\}$ of real non –zero eigen values with $|\lambda_1| \geq |\lambda_2| \geq \ldots.$ and a corresponding orthonormal sequence $\{e_n\}^{\infty}_{n=1}$ of eigen vectors s.t. $Te_n = \lambda_n e_n \; \forall \; n \geq 1$.

Now, because T is compact and $\{e_n\}$ bounded and hence bounded sequence has a convergent subsequence. ie ∃ a subsequence $1 \leq n_1 < n_2 < \ldots\ldots$ S.t $<Te_{nj}>$ converges in H as $j \to \infty$.

$\| Te_{nj} - Te_{nj'} \| \to 0$ as $j, j' \to \infty$ ………………… (1)

But $Te_{nj} = \lambda_{nj} e_{nj}$ and these vectors are mutually orthogonal for distinct j's. Hence by Pythagoras formula,. we have ∀ j< j'

$\| Te_{nj} - Te_{nj} \|^2 = \| \lambda_{nj} e_{nj} \|^2 + \| \lambda_{nj} e_{nj'} \|^2$

$= \| \lambda_{nj} \|^2 + \| \lambda_{nj'} \|^2$…………………………………………………(2)

Combining (1) and (2) Lt $|\lambda_{nj}|^2 = 0$ as $j \to \infty$,

since $|\lambda_1| \geq |\lambda_2| > \ldots\ldots$

∴ full sequence $< \lambda_n >$ converges to '0' which is third condition of the theorem.

Now, we are left to prove 2nd condition i.e.

$N(T) = \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$, Firstly, Let $x \in N(T) \Rightarrow Tx = 0$

Now $\lambda_n < e_n, x > = <\lambda_n e_n, x> = <Te_n, x > = < e_n, Tx > = 0$

$\Rightarrow x \perp e_n \Rightarrow x \in \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$

∴ $N(T) \subset \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$ ……………………….. (3)

Conversely Let $x \in \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$

then $x \in H_n \; \forall$ n and because $\| T_{|HN} \| \leq |\lambda_n|$

we have $\| Tx \| \leq |\lambda_n| \| x \|$ and Lt $\lambda_n = 0$ as $n \to \infty$

Hence $\| Tx \| = 0$ i.e. $x \in N(T)$ Hence $\text{span}(\{e_n\}^{\infty}_{n=1})^{\perp} = N(T)$ ……………………..(4)

Combining (3) and (4),we get $N(T) = \text{span}(\{e_n\}^{\infty}_{n=1})^{\perp}$. i.e., second condition is fulfilled.

Second Proof of the Theorem:

The aim of this section is to explore the relationship between T and orthonormal vectors with the help of Riemann Stieltjes Integral .This proof uses the much more advanced spectral representation that is presenting integral in terms of projections in the light of already discussed results. Let T : H → H be compact self adjoint operator defined on Hilbert space H. By using Lemma 5, We can represent T in terms of Riemann Integral as T = $\int_m^M \lambda dE\lambda$ Where m = Inf <Tx,,x> for $\| x \| = 1$

M = Sup < Tx, x > for $\| x \| = 1$ For taking value λ = m into consideration, we take

T = $\int_{m-0}^M \lambda dE\lambda$ If we consider two real members A < 0 < B s.t A < m and

B > M then T = $\int_A^B \lambda dE_\lambda$ ……………. (1)

Let $Y_\lambda = E_\lambda$ (H) which is closed subspace of H and E_λ is projection of H onto Y_λ Now, we will try to prove that dim $Y_{\lambda 0}^{\perp}$ is finite for any $\lambda_0 > 0$ Let us suppose the contrary i.e Let dim $Y_{\lambda 0}^{\perp} = \infty$. Now, we can have two options : Either $\lambda \le \lambda_0$ or $\lambda \ge \lambda_0$ Firstly if $\lambda \le \lambda_0$ then $E_\lambda E_{\lambda 0} = E_\lambda$

and if $\lambda \ge \lambda_0$ then $E_\lambda E_{\lambda 0} = E_{\lambda 0}$ (Definition of spectral family) Multiplying both sides of (1) by $E_{\lambda 0}$ i.e. T $E_{\lambda 0}$ = T = $\left(\int_A^B \lambda dE_\lambda\right) E_{\lambda 0} = \int_A^\lambda \lambda dE_\lambda$ ………. (2)

($\therefore$ Here $\lambda \le \lambda_0 \Rightarrow E_\lambda E_{\lambda 0} = E_\lambda$) Substract (2) from (1)

$T - TE_{\lambda 0} = \int_{\lambda 0}^B \lambda dE_\lambda$

$T(I - E_{\lambda 0}) = \int_{\lambda 0}^B \lambda dE_\lambda \ge \int_{\lambda 0}^B \lambda_0 dE_\lambda = \lambda_0 [E_B - E_{\lambda 0}] = = \lambda_0 I - E_{\lambda 0})$

($\therefore$ Here range is λ_0 to B) $\therefore$ T (I-$E_{\lambda 0}$) $\ge \lambda_0$ (I-$E_{\lambda 0}$) ……. (3)

Here $E_{\lambda 0}$ is projection of H onto $Y_{\lambda 0} \Rightarrow$ I-$E_{\lambda 0}$ is projection of H onto $Y_{\lambda 0}^{\perp}$

$\Rightarrow$ If $y \in Y_{\lambda 0}^{\perp} \Rightarrow (I-E_{\lambda 0})\, y = y$

By using (3) and partial order relation $< Ty, y> \ge <\lambda_0 y, y> = \lambda_0 \| y \|^2$

Rewriting, $\lambda_0 \| y \|^2 = <Ty, y> \le \|Ty\| \|y\|$

By Cauchy's Schwarz Inequality ,$\Rightarrow \lambda_0 \| y \| \le \|Ty\|$ i.e. $\| Ty \| \ge \lambda_0 \| y \| \quad \forall\, y \in Y_{\lambda 0}^{\perp}$

But we have supposed that dim $Y_{\lambda 0}^{\perp} = \infty$

$\therefore$ There is an infinite orthonormal sequence e_1, e_2 ... in $Y_{\lambda 0}^{\perp}$

s.t $\forall\, j \ne i$, $\| Te_j - Te_i \ge \lambda_0 \|e_j - e_i\| = \sqrt{2}\, \lambda_0$

This shows that there can't exist any subsequence $j_1 < j_2 <$..... s.t $< Te_j >$ is cauchy's sequence .But $<e_j>$ is bounded sequence in H and T is given to be compact. $\therefore$ Every bounded sequence should have a convergent subsequence. But corresponding to $<e_j>$, $<Te_j>$ is not convergent subsequence. This is contradiction to th e fact that T is compact**.** Hence, our supposition is wrong .

$\therefore$ dim $Y_{\lambda 0}^{\perp}$ is finite for each $\lambda_0 > 0$ and Let dim $Y_{\lambda 0}^{\perp} = d(\lambda)$

By def. of spectral family, d (λ) is decreasing sequence.

$\therefore$ for each $n \in Z^+$, the set $\{\lambda > 0 : d(\lambda) = n\}$ is either empty or an interval of the form $R^+ \cap (\mu,\mu')$ It means that there is an infinite sequence $B = \mu, > \mu_2 >$ .... of positive members s.t d (λ) is constant on each interval $[\mu_j,\mu_{j-1})$ and Lt μ_j =0as $j\to\infty$

By using definition of Riemann steiltjes integral, we claim that $\int_0^B \lambda dE_\lambda = \sum_{j=1}^{\infty} \mu_j \,(E\mu_j - E\mu_{j+1})$

Proof for claim:

Let $P_n = \{0, \mu_n, \mu_{n-1}, \ldots \mu_1 = B\}$ be partition of [0, B] Here $0 < \mu_n < \mu_{n-1} \ldots < \mu_2 < \mu_1 = B$

Again splitting subinterval (μ_j, μ_{j-1}) , $2 \le j \le n$ into n equal parts

By construction, $\|P_n\| \to 0$ as $n \to \infty$

By definition Lt S (P_n) = $\int_0^B \lambda dE_\lambda$ as $n\to\infty$………………. *

Now S (P_n) = $\sum_{j=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1}) = \mu_n (E\mu_n - E\mu_{n+1}) + \sum_{j=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1})$

$= \mu_n (E\mu_n - E_0) + \sum_{j=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1})$

Hence, $\| S(P_n) - \sum_{j=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1}) \| = \| \mu_n (E\mu_n - E_0) \| \le \mu_n \to 0$ as $n \to\infty$($\therefore E\mu_n - E_0$ is projection and its norm ≤ 1)

$\therefore$ By def. of Limit $Lt_{n\to\infty} S(P_n) = \sum_{j=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1})$……. **

i.e. $Lt_{n\to\infty} S(P_n) = \sum_{j=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})$ combining * and **

$\int_0^B \lambda dE_\lambda = \sum_{j=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})$ ……. (4)

i.e our claim is established . Now, in the next section, we will try to represent the above integral in terms of members of some orthonormal sequence. Consider the projection $E\mu_j - E\mu_{j+1}$, which is projection onto $Y\mu_{j+1}^{\perp} \cap Y\mu_j$

Let us name as $Z_j = Y\mu j_{+1}^{\perp} \cap Y\mu j$

Now, we will prove that for $j\ge1$, all z_1, z_2 ... are mutually orthogonal . Now for $\forall\ 1 \le j < i$

$\Rightarrow \mu_{j+1} \ge \mu_i$ and $\mu_1 > \mu_2 > \mu_3 >$…… is a decreasing sequence

$\Rightarrow Y\mu i \subset Y\mu_{j+1}$ .... *

and $Z_j = Y\mu_{j+1}^{\perp} \cap Y\mu j \quad \Rightarrow Z_j \perp Y\mu_{j+1}$

For j=i, $Z_i = Y\mu_{i+1} \cap Y\mu_i \Rightarrow Z_i \subset Y\mu_i$ ……………….... **

Again * and ** $Z_i \subset Y\mu_i \subset Y\mu_{j+1}$ and $Z_j \perp Y\mu_{j+1}$

$\therefore Z_i \subset Y\mu_{j+1}$ and $Z_j \perp Y\mu_{j+1}$ $\Rightarrow Z_i \perp Z_j\ \forall\ i , j$

$\therefore Z_1, Z_2$, ... are mutually orthogonal. We can choose an ON basis $e_1, e_2 \ldots e_{d1}$ in Z_1 , then similarly choose an ON basis in Z_2 and so on, we obtain an orthonormal sequence e_1, e_2 ... in H, which may be finite or infinite.

$\therefore$ we can write $(E\mu_j - E\mu_{j+1})x = x = \sum_{ek \in zj} <x, e_k> e_k$ for each j

From (4) ,$\int_0^B \lambda dE_\lambda = \sum_{j=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})$

$\left(\int_0^B \lambda dE_\lambda\right) = \sum_{j=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})x$

$= \sum_k \lambda_k <x, e_k> \forall\ x \in H$ (here $\lambda_k := \mu_j$)

i.e $\left(\int_0^B \lambda dE_\lambda\right) = \sum_k \lambda_k <x, e_k> e_k$

Here, by construction $\lambda_1 \geq \lambda_2 \geq$..... ie it is a decreasing sequence of positive members and if this sequence is infinite Lt $\lambda_k = 0$ as $k \to \infty$ Because $E\mu_j - E\mu_{j+1}$ is projection onto Z_j and $e_1, e_2 \ldots$ ..is ON in H.
Here $Z_j = Y^{\perp}\mu_{j+1} \cap Y\mu_j \Rightarrow Z_j \subset Y\mu_{j+1}^{\perp}$ and $Y_o \subset Y\mu_{j+1}$ ic $Y_0^{\perp} \supset Y\mu_{j+1}^{\perp}$
combing $Z_j \subset Y^{\perp}\mu_{j+1} \subset Y_o^{\perp}$
$Z_j \subset Y_o^{\perp} \ \forall j$ ie All $e_1, e_2, \ldots \in Z_j \subset Y_o^{\perp}$
Now, we will take the case when $\lambda_0 < 0$.Firstly, we will prove dim $Y_{\lambda 0} < \infty$
Let us start with T = $\int_A^B \lambda dE_\lambda$
$TE_{\lambda 0} = \int_A^B \lambda dE_\lambda E_{\lambda 0} = \int_A^{\lambda 0} \lambda dE_\lambda$ [Because $\lambda \leq \lambda_0$ $E_\lambda E_{\lambda 0} = E_\lambda$
$\leq \int_A^{\lambda 0} \lambda dE_\lambda = \lambda_0 [E_{\lambda 0} - E_A] = \lambda_0 E_{\lambda 0}$
That is $TE_{\lambda 0} \leq \lambda_0 E_{\lambda 0}$ Let $y \in Y_{\lambda 0}$
Now $< Ty, y> \leq < \lambda_0 y, y> = \lambda_0 \| y \|^2 < 0$ ($\therefore \lambda_0 < 0$) ..(5)
Again $| < Ty, y > | \leq \|Ty\| \|y\|$ ……………(6)
(5) and (6) $\lambda_0 \| y \|^2 \leq \| Ty \| \| y \|$ ie $\lambda_0 \| y \| \leq \| Ty \|$
ie $\| Ty \| \geq \lambda_0 \| y \| \ \forall y \in Y_{\lambda 0}$, but we suppose
dim $Y_{\lambda 0} = \infty$ $\therefore$ There is an infinite orthonormal sequence f_1, f_2 ... in $Y_{\lambda 0}$ such that
$\|T_{fj} - T_{fi}\| \geq \lambda_0 \| f_j - f_i \| = \sqrt{2}\,\lambda_0$
This shows that there can't exist any subsequence such that $< T_{fj} >$ is cauchy's sequence. But $< f_j>$ in bounded sequence in H and T is given to be compact.
$\therefore$Every bounded sequence should have a convergent subsequence. But corresponding to $< f_j >$, $< T_{fj}>$ is not convergent subsequence. This is contradiction to the fact that T is compact. Hence our supposition is wrong. $\therefore$ dim $Y_{\lambda 0}$ is finite for $\lambda_0 < 0$. and the set of all these λ's is either empty or an interval of the form $[\nu, \nu')$. It follows that there is an infinite sequence $A = \nu_1 < \nu_2 < \ldots$. of negative numbers such that Lt $\nu_j = 0$ as $j \to \infty$
Now, by using definition of Riemann Steiltjes Integral, we claim that $\int_A^0 \lambda dE_\lambda = \sum_{j=2}^{\infty} \nu_j [E\nu_j - E\nu_{j-1}]$
Let $p'_n = \{ A = \nu_1, \nu_2 \ldots \nu_n = 0 \}$ be partition of [A, 0]
By the construction as before, $\|P'_n\| \to 0$ as $n \to \infty$
By definition Lt $_{n\to\infty}S(P'_n) = \int_A^0 \lambda dE_\lambda$ ……………… (7)
Now $S(P'_n) = \sum_{j=2}^{\infty} \nu_j [E\nu_j - E\nu_{j-1}] =$ $S(P'_n) = \nu_n [E\nu_n - E\nu_{n-1}] + \sum_{j=2}^{n-1} \nu_j [E\nu_j - E\nu_{j-1}]$
$S(P'_n) = \nu_n [E_0 - E\nu_{n-1}] + \sum_{j=2}^{n-1} \nu_j [E\nu_j - E\nu_{j-1}]$
$\|S(P'_n) - \sum_{j=2}^{n-1} \nu_j [E\nu_j - E\nu_{j-1}]\| = \|\nu_n [E_0 - E\nu_n]\| \leq \nu_n \to 0$
as $n \to \infty$ (since $E_o - E\nu_n$ is projection)
$\therefore$ By def. of limit, Lt $_{n\to\infty}$ $S(P'_n) = \sum_{j=2}^{n-1} \nu_j [E\nu_j - E\nu_{j-1}]$
ie. Lt $_{n\to\infty}$ $S(P'_n) = \sum_{j=2}^{\infty} \nu_j [E\nu_j - E\nu_{j-1}]$ …………… (8)
Combining (7) and (8) $\int_A^0 \lambda dE_\lambda = \sum_{j=2}^{\infty} \nu_j [E\nu_j - E\nu_{j-1}]$
Now, in the next section, we will try to represent the above integral in terms of members of some orthonormal sequence.
Now, $E\nu_j - E\nu_{j-1}$ is projection of H onto $Y^{\perp}\nu_{j+1} \cap Y\nu_j$
let us name $Q_j = Y\nu_{j+1}^{\perp} \cap Y\nu_j$
Now, we will prove that $j \geq 2$, All Q_1, Q_2 ... are mutually orthogonal.
For $2 \leq j < i$, $\Rightarrow Y\nu i \subset Y\nu_{j-1}$
Now $Q_i = Y\nu_{i-1}^{\perp} \cap Y\nu_I$,It means $Qi \subset Y\nu_i \subset Y\nu_{j-1}$
ic $Qi \subset Y\nu_{j-1}$ and because $Q_j \perp Y\nu_{j-1}$
combining $Q_i \perp Q_j \ \forall$ i, j $\therefore Q_1, Q_2$ ... are mutually orthogonal.
$\therefore$ We can choose an orthonormal basis f_1, f_2 ... in H (by combining all ON basis in Q_1, Q_2...) which may be finite or infinite.
we can write $[E\nu_j - E\nu_{j-1}]$ x= x = $\sum_{x\in H}$ $< x, f_k > f_k$ for each k.
Now, considering $\int_A^0 \lambda dE_\lambda = \sum_{j=2}^{\infty} \nu_j [E\nu_j - E\nu_{j-1}]x$
$\left(\int_A^0 \lambda dE_\lambda\right)x = \sum_{j=2}^{\infty} \nu_j [E\nu_j - E\nu_{j-1}]x$ i.e $\left(\int_A^0 \lambda dE_\lambda\right)x = \sum_k \lambda_k' < x, f_k > f_k$, here $\nu_j := \lambda'_k$
Now, concluding the above discussion, we have derived the following results i.e.
$\left(\int_A^0 \lambda dE_\lambda\right)x = \sum_k \lambda_k' < x, f_k > f_k$ and $\left(\int_0^B \lambda dE_\lambda\right)x = \sum_k \lambda_k < x, e_k > e_k$
Combining , $Tx = \left(\int_A^B \lambda dE_\lambda\right)x = \left(\int_A^0 \lambda dE_\lambda\right)x + \left(\int_0^B \lambda dE_\lambda\right)x$
$= \sum_k \lambda_k < x, e_k > e_k + \sum_k \lambda_k' < x, f_k > f_k$
Here $e_k \perp f_k \ \forall$ k. and $e_k \in Y_o^{\perp}$ and $f_k \in Y_o$
$\therefore$ Any merging of the sequence $<e_k>$ and $<f_k>$ is a new orthonormal sequence. Hence, it follows $Te_k = \lambda_k e_k \ \forall e_k$ and $Te_k = \lambda_k f_k \ \forall f_k$ which is last part of theorem.
For the second part, we have to prove *N* (T) = span $(\{en\}^{\infty}_{n=1})^{\perp}$
If $x \in N(T) \Rightarrow Tx = 0$
$\Rightarrow < x, e_k > = 0$ and $<x, f_k > = 0$

$\Rightarrow x \perp e_k$ and $x \perp f_k$
$\Rightarrow x \in \text{span}(\{e_k\} \cup \{f_k\})^{\perp}$
Hence N (T) $\subset \text{span}(\{e_k\} \cup \{f_k\})^{\perp}$ and similarly $\text{span}(\{e_k\} \cup \{f_k\})^{\perp} \subset N(T)$
combining , $N(T) = \text{span}(\{e_k\} \cup \{f_k\})^{\perp}$ which is required result. Hence, all claims of theorem are fulfilled

III. CONCLUSION

We basically discussed the spectral representation of compact self adjoint operators. The significance and usefulness of this result lies in the fact that we can represent T (ν),ν ∈ H in a simple and unique form. Sometimes it is possible and convenient to break up a vector space into special disjoint subspaces. So, we are writing H as, $H = Y + Y^{\perp}$, where we assume $Y = \text{span}(\{e_n\}^{N}_{n=1})$ We then proved kernel of T as orthogonal of the set which we proved as combination of the orthonormal elements, that is , we presented Hilbert space as $H = Y + N(T)$. This process is done for both the cases , that is when N (number of elements in the orthonormal sequence) is finite and other when N is infinite. This made study of infinite dimensional spaces both richer as well as crucial. In the second proof, we expressed T in terms of projections and then using some very important results ,we presented T in terms of Riemann Steiltjes Integral. That Integral was written in terms of some orthonormal sequence for required result. Motivated for the application of this theorem, we here tried to write ν Є H in the form $\nu = (\sum \alpha_n e_n) + Z$, Where $\alpha_n \in k$, $Z \in N(T)$ and for each such vector ν,we have $T(\nu) = \sum \lambda_n \alpha_n e_n$ which is required representation and this presentation is unique.

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