International Journal of Emerging Research in Management &Technology
ISSN: 2278-9359 (Volume-4, Issue-6)
Research Article **June 2015**

Fixed Point Theorem in Chatterjea Mapping

Komal Goyal*
Department of Mathematics
Jaypee University
India

Amita Garg
Department of Mathematics
Sharda University
India

Abstract—

The purpose of this paper is to apply fixed point theorem in Chatterjea mapping. We verify the theorem using an example also. Our main theorem extends and improves fixed point theorem in literature.

Keywords— ***Fixed point theorem, Chatterjea Mapping, Fixed point, Metric space.***

I. INTRODUCTION

Fixed point theory is one of the most dynamic research subjects in nonlinear science. The feasibility of application of it to the various disciplines increases its beauty more and more. The most impressive, smart technique to find the desired fixed point was given by Banach which gave direction to several authors in solving various problems in different research field. Thereafter, many authors proved different generalizations by proving different types of fixed point theorems in complete metric space like Reich [11],Kannan [5],Suzuki [18] , Hardy G.E. and Rogers [4]. Following this trend, in 1972, Chatterjea [2] introduced the following definition.

A mapping T on a metric space (X,d) is called Chatterjea if there exists $\alpha \epsilon [0,\frac{1}{2})$ such that

$d(Tx,Ty) \leq \alpha d(x,Ty) + \beta d(y,Tx)$ for all $x, y \in X$.

Chatterjea contraction is similar to Kannan contraction. Motivated by the Kannan, we prove a fixed point theorem for Chatterjea mapping. This paper opens way to apply this fixed point theorem for some more different settings given by Rhoades [13].

The article includes the example showing that the obtained result is significant.

II. LEMMA

See [18,3] : Let (X, d) be a metric space and let T be a mapping on X. Let $x \in X$ satisfy $d(Tx,Ty) \leq rd(x,y)$ forall $x,y \in X$ for some $r \in [0,1)$. Then, $(1+r)^{-1}d(x,Tx) \leq d(x,y)$ holds.

III. FIXED POINT THEOREM

We define Δ as given below:

We first put Δ and $\Delta_{j.}(j = 1,\ldots,4)$ by

$\Delta = \{(\alpha,\beta): \alpha \geq 0, \beta \geq 0, \alpha + \beta < 1,$

$\Delta_1 = \{(\alpha,\beta) \in \Delta: \alpha \leq \beta, \alpha + \beta + \alpha^2 < 1,$

$\Delta_2 = \{(\alpha,\beta) \in \Delta: \alpha \geq \beta, \alpha + \beta + \beta^2 < 1,$

$\Delta_3 = \{(\alpha,\beta) \in \Delta: \alpha \geq \beta, \alpha + \beta + \alpha^2 \geq 1,$

$$\Delta_4 = \{(\alpha,\beta) \in \Delta: \alpha \leq \beta, \alpha + \beta + \alpha^2 \geq 1, \quad (4.1)$$

IV. MAIN THEOREM

See [3] Define a non increasing function ψ from Δ into $(1/2, 1]$ by

$$\varphi(\alpha,\beta) = \begin{cases} 1 & if\ (\alpha,\beta) \in \Delta_1 \\ 1 & if\ (\alpha,\beta) \in \Delta_2 \\ 1-\beta & if\ (\alpha,\beta) \in \Delta_3 \\ \frac{1-\beta}{1-\beta+\alpha} & if\ (\alpha,\beta) \in \Delta_4 \end{cases} \quad (4.2)$$

Let T be a mapping on a complete metric space (X, d). Assume that there exists $\alpha, \beta \in \Delta$ such that

$$\varphi(\alpha,\beta)d(x,Ty) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq \alpha d(x,Ty) + \beta d(y,Tx). \quad (4.3)$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover $\lim_n T^n x = z$ holds for every $x \in X$.

Proof: We put

$$q = \frac{\beta}{1-\alpha} \in [0,1) \quad and \quad r = \frac{\alpha}{1-\beta} \in [0,1) \quad (4.4)$$

Since $\varphi(\alpha,\beta) \le 1$,
$\varphi(\alpha,\beta)d(x,Ty) \le d(x,Ty)$ holds.
From assumptions, we have

$$d(Ty,T^2y) \le \alpha d(x,Ty) + \beta d(Ty,T^2y) \qquad (4.5)$$

and hence

$$d(Ty,T^2y) \le rd(x,Ty) \text{ for all } x \in X \qquad (4.6)$$

Since,

$$\varphi(\alpha,\beta)d(Ty,T^2y) \le d(Ty,T^2y) \le rd(x,Ty) \le d(x,Ty) \qquad (4.7)$$

We have,

$$d(T^2y,Ty) \le \alpha d(Ty,T^2y) + \beta d(x,Ty) \qquad (4.8)$$

and hence

$$d(Ty,T^2y) \le qd(x,Ty) \text{ for all } x \in X \qquad (4.9)$$

Fix $u \in X$ and put $u_n = T^n u$ for $n \in N$.
From (4.6), we have

$$\sum_{n=1}^{\infty} d(u_n,u_{n+1}) \le \sum_{n=1}^{\infty} r^n d(u,Tu) < \infty \qquad (4.10)$$

So, $\{u_n\}$ is a Cauchy sequence in *X*. Since *X* is complete, $\{u_n\}$ converges to some point $z \in X$.
We next show

$$d(z,Ty) \le \beta d(x,Ty) \text{ for all } x \in X/\{z\} \qquad (4.11)$$

Since $\{u_n\}$ converges, for sufficiently large $n \in N$,
We have

$$\varphi(\alpha,\beta)d(u_n,Tu_n) \le d(u_n,u_{n+1}) \le d(u_n,x) \qquad (4.12)$$

And hence

$$d(Tu_n,Ty) \le \alpha d(u_n,Tu_n) + \beta d(x,Ty) \qquad (4.13)$$

Therefore, we obtain

$$d(z,Ty) = \lim_{n\to\infty} d(u_{n+1},Ty) = \lim_{n\to\infty} d(Tu_n,Ty) \le \lim_{n\to\infty}\{\alpha d(u_n,Tu_n) + \beta d(x,Ty)\} = \beta d(x,Ty) \text{ for all } x \in X/\{z\} \qquad (4.14)$$

By (4.11) we have

$$d(x,Ty) \le d(x,z) + d(z,Ty) \le d(x,z) + \beta d(x,Ty) \qquad (4.15)$$

And hence

$$(1-\beta)d(x,Ty) \le d(x,z) \text{ for all } x \in X/\{z\} \qquad (4.16)$$

Let us prove that *z* is a fixed point of *T*.

(i) Where $(\alpha,\beta) \epsilon \Delta_1$, we assume $Tz \ne z$.

Then we have,

$$d(Tz,T^2z) \le rd(z,Tz) < d(z,Tz) = \lim_{n\to\infty} d(Tz,u_n) \qquad (4.17)$$

So for sufficiently large $n \in N$,

$$\varphi(\alpha,\beta)d(Tz,T^2z) = d(Tz,T^2z) \le d(Tz,u_n) \qquad (4.18)$$

holds and hence

$$d(T^2z,z) = \lim_{n\to\infty} d(T^2z,Tu_n) \le \lim_{n\to\infty}(\alpha d(Tz,T^2z) + \beta d(u_n,Tu_n)) = \alpha d(Tz,T^2z) \qquad (4.19)$$

Thus we obtain,

$$d(z,Tz) \le d(z,T^2z) + d(Tz,T^2z) \le (1+\alpha)d(Tz,T^2z) \le (1+\alpha)rd(z,Tz) = \frac{\alpha+\alpha^2}{1-\beta}d(z,Tz) < d(z,Tz) \qquad (4.20)$$

Which is a contradiction. Therefore, we obtain $Tz = z$.

(ii) Where $(\alpha,\beta) \epsilon \Delta_2$, we assume $Tz \ne z$.

We have,

$$d(z,Tz) \le d(z,T^2z) + d(Tz,T^2z) \le (1+\beta)d(Tz,T^2z) \le (1+\beta)qd(z,Tz) = \frac{\beta+\beta^2}{1-\alpha}d(z,Tz) < d(z,Tz) \qquad (4.21)$$

Which is a contradiction. Therefore, we obtain $Tz = z$.

Where $(\alpha,\beta) \epsilon \Delta_3$, we consider two following cases here.

(a) There exist at least two natural number n satisfying $u_n = z$.

(b) $u_n \neq z$ for sufficiently large $n \in N$.

In the first case, if we assume $Tz \neq z$, then $\{u_n\}$ can not be Cauchy. Therefore $Tz = z$.

In second case, we have by (4.16),

$\varphi(\alpha,\beta)d(u_n,Tu_n) \leq d(u_n,z)$ for sufficiently large $n \in N$.

From the assumption,

$$d(z,Tz) = \lim_{n\to\infty} d(Tu_n,Tz) \leq \lim_{n\to\infty}\left(\alpha d(u_n,Tu_n) + \beta d(z,Tz)\right) = \beta d(z,Tz) \quad (4.22)$$

Since $\beta < 1$, we obtain $Tz = z$.

(i) Where $(\alpha,\beta)\epsilon\Delta_4$, we note that

$\varphi(\alpha,\beta) = (1+r)^{-1}$.

By lemma II, either $\varphi(\alpha,\beta)d(u_n,Tu_n) \leq d(u_n,z)$

Or $$\varphi(\alpha,\beta)d(Tu_n,T^2u_n) \leq d(Tu_n,z) \quad (4.23)$$

Holds for every $n \in N$. Thus, there exists a sub sequence $\{n_j\}$ of $\{n\}$ such that,

$$\varphi(\alpha,\beta)d\left(u_{nj},Tu_{nj}\right) \leq d\left(u_{nj},z\right) \quad (4.24)$$

For $j \in N$. From assumption, we have

$$d(z,Tz) = \lim_{j\to\infty} d(Tu_{nj},Tz) \leq \lim_{j\to\infty}(\alpha d(u_{nj},Tu_{nj}) + \beta d(z,Tz)) = \beta d(z,Tz) (4.25)$$

Since $\beta < 1$, we obtain $Tz = z$.

Therefore we have shown $Tz = z$ in all cases.

From (4.11), the fixed point z is unique.

This proves our theorem.

V. EXAMPLE

Put $X = [-15,-14] \cup \{0\} \cup [14,15]$ and define a mapping T on X by

$$Tx = \begin{cases} +\dfrac{(x)^2 - (14)^2}{1.5} & if\ x \in [-15,-14) \\ 0 & if\ x \in \{-14,0,+14\} \\ -\dfrac{(x)^2 - (14)^2}{1.5} & if\ x \in (+15,+14] \end{cases}$$

For $x \in X$. Then T satisfies the main theorem.

VI. CONCLUSION

It concludes if it satisfies this fixed point theorem in Chatterjea Mapping, then several other theorems can be proved in Chatterjea Mapping.

REFERENCES

[1] D.W.Boyd, Wong, *On nonlinear contractions*, Proc. Am. Math. Soc. 20 (1969), pp.458-464.

[2] S.K.Chatterjea, *Fixed point theorem*, C. R. Acad. Bulgare Sci. 25 (1972), pp.727-730.

[3] Y.Enjouji, M.Nakanishi, T.Suzuki, *A Generalization of Kannan's Fixed Point Theorem*, Fixed Point Theory and Applications Volume 2009 (2009), pp.1-10.

[4] G.E.Hardy and T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad Math. Bull. 16 (1973), pp.201-206.

[5] R.Kannan, *Some results on fixed point,* Bull. Calcutta Math.Soc. 60(1968), pp.71-76.

[6] R.Kannan, *Some results on fixed points*. II, American Mathematical Monthly, vol. 76 (1969), pp. 405–408.

[7] E.Karapinar, *Edelstein type fixed point theorem*, Fixed Point Theory and Applications *(*2012), pp.1-12.

[8] M.Kikkawa, T.Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*, Nonlinear Anal. 69 (2008), pp.2942–2949.

[9] M.Kikkawa, T.Suzuki, *Some similarity between contractions and Kannan mappings*, Fixed Point Theory Appl. 2008 (2008), Article ID 649749, pp.1–8.

[10] S. Reich, *Kannan's fixed point theorem*, Boll. Unione Mat. Ltal. 4(4) (1971), pp.1-11.

[11] S.Reich , *Some fixed point problems,* Atti .Accad .Naz. Lincei 57(1974), pp.194-198.

[12] S.Reich, Some fixed point problems. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. 57 (1975), pp.194-198.

[13] B.E.Rhoades, *A comparison of various definition of contractive mapping*, Transaction of the American Mathematical Society, Vol.226 (1977), pp.257-290.

[14] B.E. Rhoades, *A collection of contractive definition,* Mathematics Seminar Notes Vol.6 (1978), pp.1-10.

[15] B.E.Rhoades, *Some theorems on weakly contractive maps,* Nonlinear Anal. 47 (2001), pp.2683-2693.

[16] P. V. Subrahmanyam, *Completeness and fixed-points*, Monatshefte fu¨ r Mathematik, vol. 80, no. 4 (1975), pp.325–330.

[17] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc. 136 (2008), pp.1861-1865.

[18] T.Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Anal.Theory Methods Appl. 71(11) (2009), pp.5313-531.