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Research Article

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Stability of Equilibria in a Discrete Model of Interacting Species

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Abstract—

This paper discusses the stability of equilibrium points of a discrete prey-predator system with three species. The dynamical analysis of the system is performed with numerical simulation which supports the theoretical findings. Time series plots are obtained for different sets of parameter values. Also bifurcation diagrams are plotted to show complex and rich dynamical behaviour of the system in selected range of growth parameter.

Keywords— ***Equilibrium points, prey-predator system, difference equations, bifurcation***

I. INTRODUCTION

Mathematical properties and ecological meaning of continuous and discrete models have been investigated qualitatively and numerically in order to explain mutual interactions between populations. Prey-predator models are of great interest to both ecologists and mathematicians, because the problem attempts to model the complex relationship in the populations of different species that share the same environment. Theoretical and numerical studies of these models enable us to understand the interactions of biological populations [6]. In recent decades, many researchers [2, 5, 8, 10] have focused on the ecological models with three and more species to understand complex dynamical behaviours of ecological systems in the real world. The models have demonstrated very complex dynamic nature of the interactions. In recent years, the Lotka-Volterra predator-prey interactions modelled by system of difference equations have been proposed and studied extensively. The discrete time models produce much richer patterns [1, 4, 7, 9, 11, 12] and are ideally suited to describe the population dynamics of species, which are characterized by discrete generations.

II. FORMULATION OF THE MODEL

In this paper, we consider the discrete-time prey-predator system describing the interactions among three species by the following system of difference equations:

$$\begin{aligned} x(t+1) &= ax(t)(1-x(t)) - x(t)z(t) + x(t)y(t)z(t) \\ y(t+1) &= by(t)(1-y(t)) - y(t)z(t) + x(t)y(t)z(t) \\ z(t+1) &= z(t)(1-cz(t)) + d\,x(t)z(t) + e\,y(t)z(t) \end{aligned} \tag{1}$$

where the densities of prey species are denoted by x(t), y(t) and z(t). It is assumed that all the species in the model grow logistically. Also the prey species help each other against predator. This is a discrete version of a model discussed in [3].

III. EQUILIBRIA OF THE MODE

The equilibrium points of (1) are the solution of the following equations

$x = ax(1-x) - xz + xyz$; $y = by(1-y) - yz + xyz$; $z = z(1-cz) + dxz + eyz$.

Solving the above set of simultaneous equations, we obtain the equilibrium points $E_0(0,0,0)$ (trivial), $E_1\left(0, \frac{b-1}{b}, 0\right)$, $E_2\left(\frac{a-1}{a} 0, 0\right)$, $E_3\left(\frac{a-1}{a}, \frac{b-1}{b}, 0\right)$ and $E_4\left(0, \frac{c(b-1)}{bc+e}, \frac{e(b-1)}{bc+e}\right)$ (axial).

IV. ANALYSIS OF DYNAMICS OF THE MODEL AND NUMERICAL EXAMPLES

In this section, we analyse the stability of the system (1) around equilibrium points. The stability analysis of the system (1) is performed by computing the Jacobian matrix corresponding to each equilibrium point. The Jacobian matrix J for the linearized system (1) is

$$J(x,y,z) = \begin{bmatrix} a(1-2x)+z(y-1) & xz & x(y-1) \\ yz & b(1-2y)+z(x-1) & y(x-1) \\ dz & ez & 1-2cz+dz+ey \end{bmatrix} \tag{2}$$

Since we are interested in the nontrivial equilibrium points, we neglect E_0 . Time plots and phase portraits are presented.

Theorem 1. The equilibrium point E_1 is locally asymptotically stable if $a < 1$, $1 < b < 3$ and $0 < c < \frac{2b}{1-b}$, otherwise unstable equilibrium point.

Proof: The Jacobian matrix J evaluated at the equilibrium point E_1 is

$$J\left(E_1\right)=\begin{bmatrix} a & 0 & 0 \\ 0 & 2-b & \dfrac{1-b}{b} \\ 0 & 0 & 1+\dfrac{e\left(b-1\right)}{b} \end{bmatrix}$$

hence the eigenvalues of the matrix $J\left(E_1\right)$ are $\lambda_1 = a$, $\lambda_2 = 2-b$ and $\lambda_3 = 1+\dfrac{e\left(b-1\right)}{b}$. Hence E_1 is locally asymptotically stable if $a<1$, $1<b<3$ and $0<c<\dfrac{2b}{1-b}$, and otherwise unstable $a>1$, $b>3$ and $e>\dfrac{2b}{1-b}$. The time plot trajectory and phase portrait are provided below to exhibit stability around E_1.

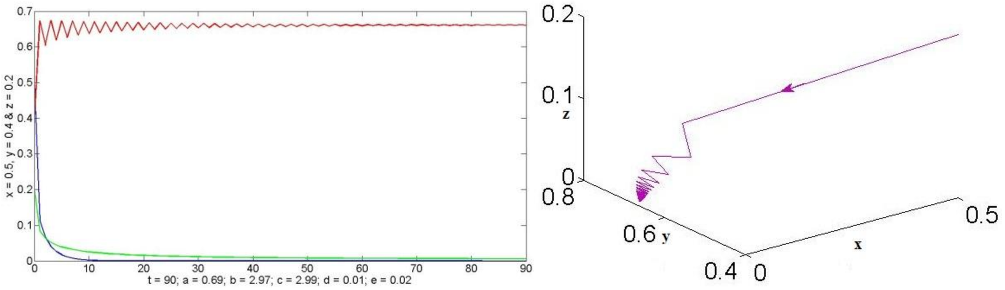

Fig. 1. Stability at E_1.

Theorem 2. The equilibrium point E_2 is locally asymptotically stable if $1<a<3$, $b<1$ and $0<d<\dfrac{2a}{1-a}$, otherwise unstable equilibrium point.

Proof: At the equilibrium point E_2 The Jacobian matrix J for the system is

$$J\left(E_2\right)=\begin{bmatrix} 2-a & 0 & 1-a \\ 0 & b & 0 \\ 0 & 0 & 1+\dfrac{d\left(a-1\right)}{a} \end{bmatrix}$$

Hence the eigenvalues of the matrix $J\left(E_2\right)$ are $\lambda_1 = 2-a$, $\lambda_2 = b$ and $\lambda_3 = 1+\dfrac{d\left(a-1\right)}{a}$. Hence E_2 is locally asymptotically stable if $1<a<3$, $b<1$ and $0<d<\dfrac{2a}{1-a}$, and otherwise unstable $a>3$, $b>1$ and $d>\dfrac{2a}{1-a}$. Stability around the equilibrium point E_2 is illustrated in the Figure – 2.

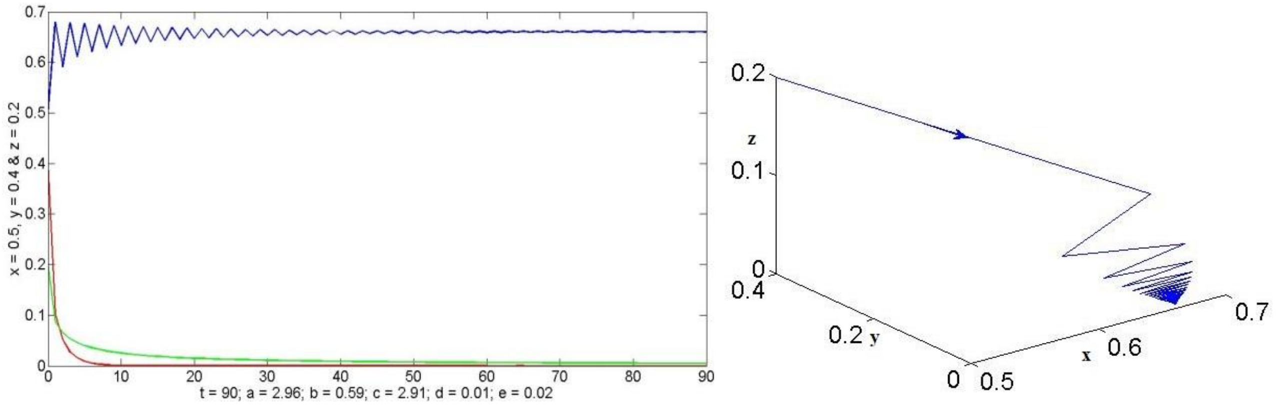

Fig. 2. Stability at E_2.

Theorem 3. The equilibrium point E3 is locally asymptotically stable if $1<a<3$, $1<b<3$ and $\dfrac{ae\left(b-1\right)}{b\left(1-a\right)}<d<\dfrac{a}{1-a}\left(2+\dfrac{e\left(b-1\right)}{b}\right)$, otherwise unstable equilibrium point.

Proof: The Jacobian matrix evaluated at E_3 is of the form

$$J(E_3)=\begin{bmatrix} 2-a & 0 & \frac{(1-a)}{ab} \\ 0 & 2-b & \frac{(1-b)}{ab} \\ 0 & 0 & 1+\frac{d(a-1)}{a}+\frac{e(b-1)}{b} \end{bmatrix}$$

Hence the eigenvalues of the matrix $J(E_3)$ are $\lambda_1 = 2-a$, $\lambda_2 = 2-b$ and $\lambda_3 = 1+\frac{d(a-1)}{a}+\frac{e(b-1)}{b}$. Hence E_3 is locally asymptotically stable when $1<a<3$, $1<b<3$ and $\frac{ae(b-1)}{b(1-a)}<d<\frac{a}{1-a}\left(2+\frac{e(b-1)}{b}\right)$, and unstable when $a>3$, $b>3$ and $\frac{ae(b-1)}{b(1-a)}>d$ or $d>\frac{a}{1-a}\left(2+\frac{e(b-1)}{b}\right)$.

Figure – 3 illustrates the stability at E_3.

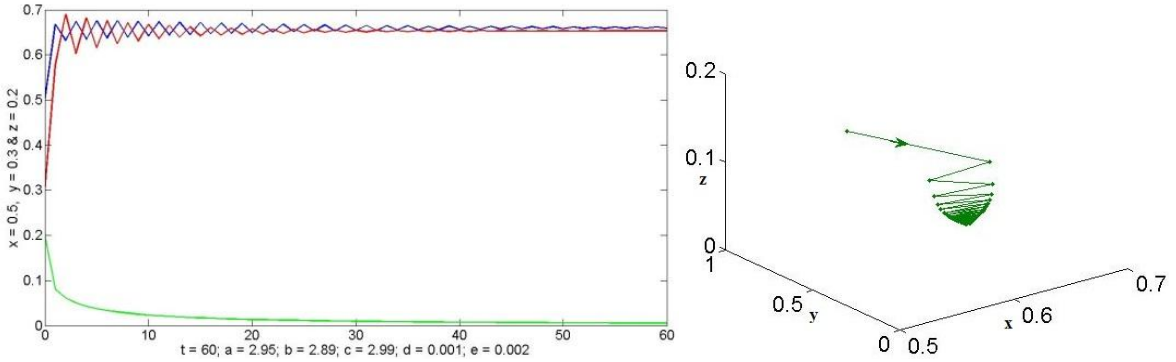

Fig. 3. Stability at E_3.

Theorem 4. The equilibrium point E_4 is locally asymptotically stable if and only if A, C and $AB-C$ are positive.

Proof: The Jacobian matrix J at E_4 has the form

$$J(E_4)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (3)$$

where, $a_{11} = a-\frac{e(b-1)}{bc+e}+\frac{ce(b-1)^2}{(bc+e)^2}$, $a_{12}=0$, $a_{13}=0$, $a_{21}=\frac{ce(b-1)^2}{(bc+e)^2}$, $a_{22}=\frac{bc(2-b)+e}{bc+e}$, $a_{23}=\frac{c(1-b)}{bc+e}$, $a_{31}=\frac{de(b-1)}{bc+e}$, $a_{32}=\frac{e^2(b-1)}{bc+e}$, $a_{33}=1-\frac{ce(b-1)}{bc+e}$.

The characteristic equation is

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (4)$$

with $A=-(a_{11}+a_{22}+a_{33})$, $B=a_{11}a_{33}+a_{22}a_{33}+a_{11}a_{22}-a_{23}a_{32}$ and $C=(a_{23}a_{32}-a_{22}a_{33})a_{11}$. By the Routh-Hurwitz criterion, E_4 is locally asymptotically stable if and only if A, C, and $AB-C$ are positive.

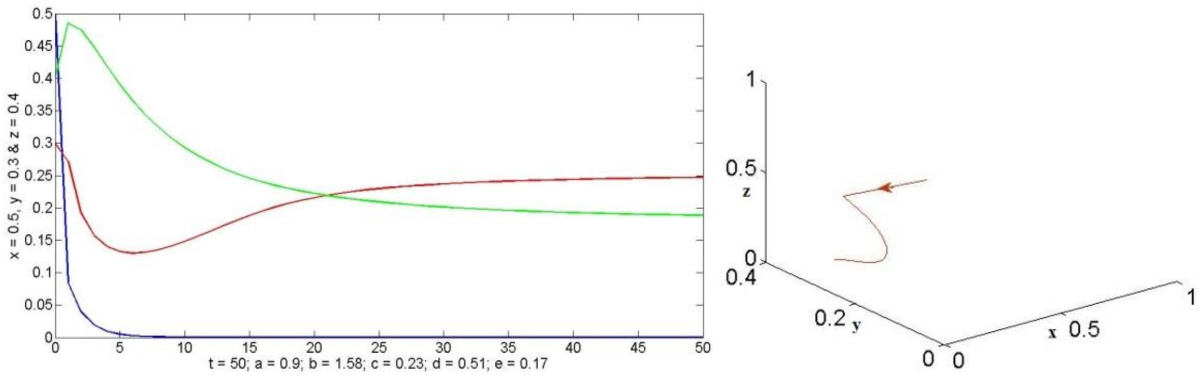

Fig. 4.Time Series Plot and Phase Portrait at E_4.

V. BIFURCATION DIAGRAMS

Bifurcation is a change of the dynamical behaviours of the system as its parameters pass through a bifurcation (critical) value. Bifurcation usually occurs when the stability of an equilibrium changes. An examination of the bifurcation diagram of the system shows that a given qualitative change in the functioning of the ecosystem may be produced by changing different parameters. We restrict our analysis to events that occur when the parameter a is changed, and the other parameters are fixed at given values. In this section, we focus on exploring the possibility of chaotic behaviour

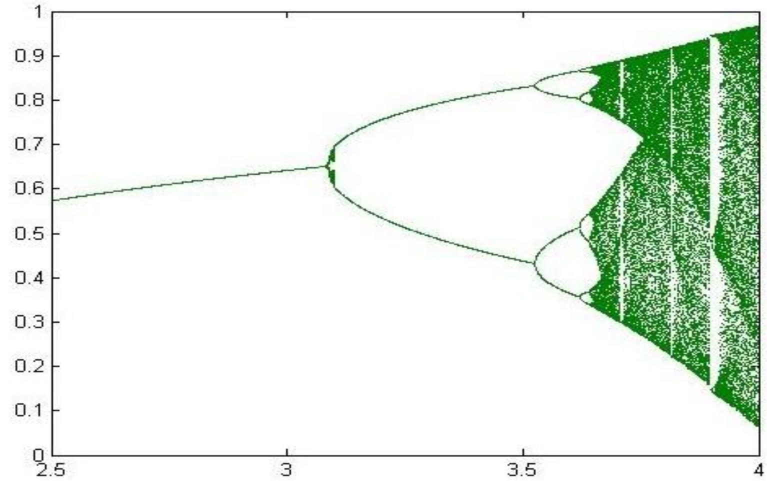

Fig. 5. Bifurcation diagram for prey population

for prey densities of the system (1) with initial conditions x = 0.6, y = 0.3 and $z = 0.4$ with parameter values b = 0.25, c = 1.28, d = 0.15, e = 0.26 and a varies from 2.5 to 4 with step 0.001. It can be observed that the first bifurcation value of the system (1) is between the numbers 2.5 and 3.1, and the second one is between 3.5 and 3.6, and so on.

Finally in Figure – 6, one of the most striking properties of the prey – predator system described by system (1) is the diversity of behaviours that are obtained for different values of the intrinsic growth rate a. Figure 6 shows few dynamical behaviours.

Note that in panels (b) and (c) of this figure the growth pattern becomes periodic after an initial condition. The period is two samples in panel (b), and four samples in panel (c). No obvious pattern is discernible in panel (d), a regime that is usually referred to as chaos.

The numerical simulation in this section is consistent with the analytical results obtained in the former sections and supports the mathematical analysis.

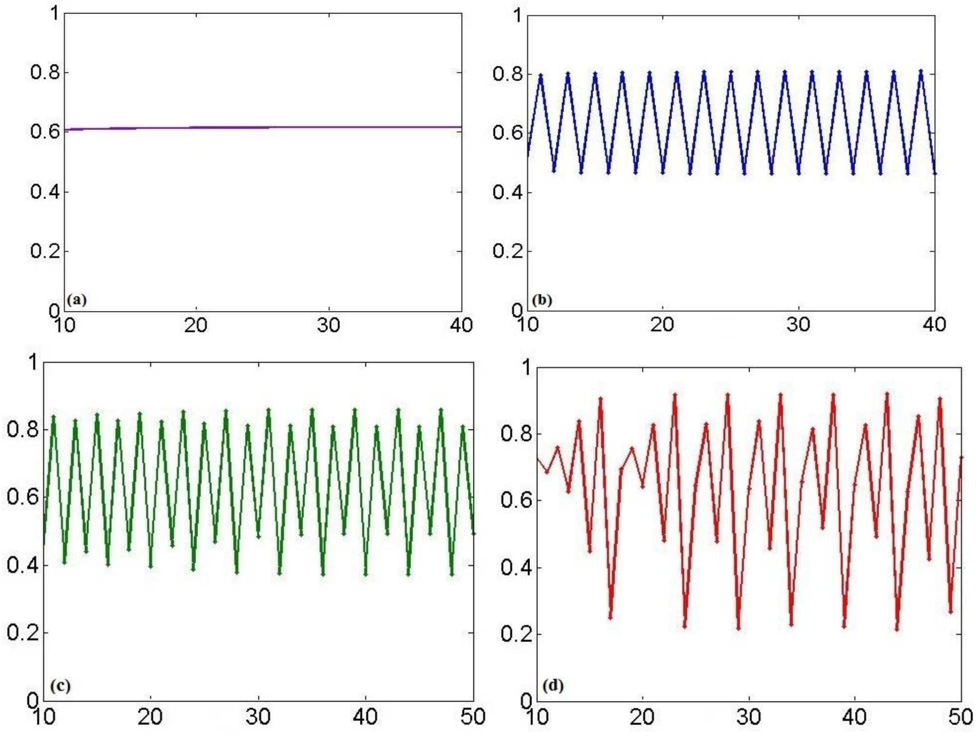

Fig. 6.Patterns of population growth with initial condition $x_0 = 0.6$ and for the following values of intrinsic growth rates a: (a) 2.8, steady state; (b) 3.4, two-value cycle; (c) 3.58, four-value cycle; (d) 3.829, chaos.

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