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Some Fixed Point Theorems Using n-property in Menger Space

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Abstract-

The purpose of this paper is to prove a coupled fixed point theorem for contractive mappings in complete Menger space satisfying n-property.

Keywords - Probabilistic metric space, coupled fixed point, semi-compatible, coupled coincidence point and n-property.

I. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by the great mathematician Menger [6] who used distribution functions instead of nonnegative real numbers as the value of metric. Later there are many authors who have some detailed discussions and applications of a probabilistic metric space, for example, we may see Schweizer and Sklar [10]. Besides, there are many results about fixed point theorems in probabilistic metric space with contractive types having appeared.The notion of compatible maps in Menger Spaces has been introduced by Mishra [9]. In this paper, using the idea of compatible maps. we apply semi-compatibility for pair of mappings in Menger space and here we prove a coupled fixed point theorem for contractive mappings in complete Menger space satisfying n-property. Which was introduced in fuzzy metric space by sedghi *et al* [16].

II. PRELIMINARIES

In this section, we recall some definitions and known results in Menger Probabilistic metric space. For more details, we refer the readers to [1-9,12].

Definition 2.1 A triangular norm $*$ (shortly t-norm) is a binary operation on the unit interval [0, 1] such that for all $a, b, c, d \in [0,1]$ the following conditions are satisfied :

(a) $a * 1 = a$;

(b) $a * b = b * a$;

(c) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;

(d) $a * (b * c) = (a * b) * c$.

Example 2.1 $a * b = \max\{a + b - 1,0\}$ and $a * b = \min\{a, b\}$.

Definition 2.2 A distribution function is a function $F: [-\infty, \infty] \to [0,1]$ which is left continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0, F(\infty) = 1$.

We will denote $\triangle$ the family of all distribution functions on $[-\infty, \infty]$. H is special element of $\triangle$defined by

$$H(t) = \begin{cases} 0 \ \ if\ t \leq 0, \\ 1\ if\ t > 0. \end{cases}$$

If X is nonempty set, $F: X \times X \to \triangle$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by F_{xy}.

Definition 2.3 (Schweizer and Sklar [10]) The ordered pair (X, F) is called a probabilistic metric space (shortly PM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(FM-0) $F_{xy}(t) = 1 \Leftrightarrow x = y$;

(FM-1) $F_{xy}(0) = 0$;

(FM-2) $F_{xy} = F_{yx}$

(FM-3) $F_{xz}(t) = 1, F_{zy}(s) = 1 \Longrightarrow F_{xy}(t + s) = 1$.

The ordered triple $(X, F, *)$ is called a Menger space if (X, F) is a PM-space, $*$ is a t-norm and the following conditions is also satisfied: for all $x, y, z \in X$ and $t, s > 0$,

(FM-4) $F_{xy}(t + s) \geq F_{xz}(t) * F_{zy}(s)$.

Proposition 2.4 (Sehgal and Bharucha- Reid [15]) Let (X, d) be a metric space. Then the metric d induces a distribution function F defined by

$F_{xy}(t) = H(t - d(x,y)$ for all $x, y \in X$ and $t > 0$. If t-norm $*$ is $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$ then $(X, F, *)$ is a Menger Space. Further, $(X, F, *)$ is a complete Menger space if (X, d) is complete.

Definition 2.5 (Mishra [9]) Let $(X, F, *)$ be a Menger space and $*$ be a continuous t-norm

(a) A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n \to x$) iff for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.and $p > 0$.

(c) A menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.6 If $*$ is a continuous t-norm, it follows from (FM-4) that the limit of sequence in Menger space is uniquely determined.

Definition 2.6 [16] Let $(X, F, *)$ be a Menger space, F is said to satisfies the n-property on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} [F_{xy}(k^n t)]^{n^p} = 1$$

Whenever $x, y \in X$, $k > 1$ and $p > 0$.

Example 2.7 (Example of n-property) Let (X, d) be an ordinary metric space $a * b = ab$ for all $a, b \in [0,1]$ and $F_{xy}(t) = \left[\frac{t}{t+1}\right]^{d(x,y)}$ for every $x, y \in X$, $t > 0$ then we have

$$\lim_{n\to\infty} [F_{xy}(k^n t)]^{n^p} = \lim_{n\to\infty} \left[\frac{k^n t}{k^n t + 1}\right]^{d(x,y)n^p} = e^A = 1$$

Since

$$A = \lim_{n\to\infty} \left[\frac{k^n t}{k^n t+1} - 1\right] d(x,y) n^p = \lim_{n\to\infty} \frac{-d(x,y)n^p}{k^n t+1} = 0.$$

Therefore F satisfies the n-property on $X^2 \times (0, \infty)$.

Lemma 2.8 Let $\{x_n\}$ be a sequence in a Menger space $(X, F, *)$ with continuous t-norm $*$ and $a * b \geq ab$. For all $a, b \in [0,1]$ F satisfies the n-property. Suppose $\{x_n\}$ is a sequence in X such that for all $n \in N$

$$F_{x_n x_{n+1}}(kt) \geq F_{x_{n-1} x_n}(t)$$

For every $0 < k < 1$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof- For every $x_n, x_{n+1} \in X$, we have

$F_{x_n x_{n+1}}(t) \geq F_{x_0 x_1}\left(\frac{t}{k^n}\right)$

On the other hand for every $m > n$, and $0 < k < 1$ we have

$$(1-k)(1 + k + \cdots \ldots . + k^{m-n-1}) = 1 - k^{m-n} < 1$$

And so for every $t > 0$, we have

$$t(1-k)(1 + k + \cdots \ldots . + k^{m-n-1}) < t$$

Now since F is non-decreasing therefore

$$F_{x_n x_m}(t) \geq F_{x_n x_m}(t(1-k)(1 + k + \cdots \ldots . + k^{m-n-1}))$$
$$\geq F_{x_n x_{n+1}}(t(1-k)) * F_{x_{n+1} x_{n+2}}(t(1-k)k) * \ldots F_{x_{m-1} x_m} t(1-k)k^{m-n-1}$$
$$\geq F_{x_0 x_1}\left(\frac{(1-k)t}{k^n}\right) * F_{x_0 x_1}\left(\frac{(1-k)t}{k^n}\right) * \ldots F_{x_0 x_1}\left(\frac{(1-k)t}{k^n}\right)$$
$$\geq \left[F_{x_0 x_1}\left(\frac{(1-k)t}{k^n}\right)\right]^{m-n}$$
$$\geq \left[F_{x_0 x_1}\left(\frac{(1-k)t}{k^n}\right)\right]^{m}$$
$$\geq \left[F_{x_0 x_1}\left(\frac{(1-k)t}{k^n}\right)\right]^{n^q} \to 1.$$

Where $q > 0$, such that $m \leq n^q$. Hence the sequence $\{x_n\}$ is a Cauchy sequence.

Definition 2.7 An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $T: X \times X \to X$ if

$$T(x, y) = x, T(y, x) = y$$

Definition 2.8 An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $f: X \times X \to X$ *and* $g: X \to X$ if

f(x,y) = gx , f(y,x) = gy .

Definition 2.9 The mappings $f: X \times X \to X$ and $g: X \to X$ are said to be semi-compatible if

$$\lim_{n\to\infty} F_{gf(x_n,y_n)f(x,y)}(t) = 1$$

or

$\lim_{n\to\infty} F_{gf(y_n,x_n)f(x,y)}(t) = 1$.
For all $t > 0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that
$\lim_{n\to\infty} f(x_n,y_n) = \lim_{n\to\infty} g(x_n) = x$
$\lim_{n\to\infty} f(y_n,x_n) = \lim_{n\to\infty} g(y_n) = y$
For all $x, y \in X$ are satisfied .
Note - for convenience we denote

$$[F_{xy}(t)]^n = \underbrace{F_{xy}(t) * F_{xy}(t) * \dots F_{xy}(t)}_{n-times}$$

For all $n \in N$.

III. Main Results

Theorem 3.1 Let $a * b \geq ab$, for all $a, b \in [0,1]$ and $(X, F, *)$ be a complete Menger space such that F has n-property. Let $f: X \times X \to X$ and $g: X \to X$ be two functions such that

$F_{f(x,y)f(u,v)}(kt) \geq F_{gxgu}(t) * F_{gygv}(t)$ … (2.1)

for all $x, y, u, v \in X$, where $0 < k < 1$, $f(X \times X) \subseteq g(X)$ and g is continuous and the pair $[g, f]$ is semi-compatible. Then there exists a unique $v \in X$, such that

$$x = gx = f(x,x) .$$

Proof- Let x_0, y_0 be two arbitrary points of X. Since $f(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = f(x_0, y_0)$ and $gy_1 = f(y_0, x_0)$ again from $f(X \times X) \subseteq g(X)$ we can choose $x_2, y_2 \in X$ such that $gx_2 = f(x_1, y_1)$ and $gy_2 = f(y_1, x_1)$.continuing this process we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = f(x_n, y_n) \text{ and } gy_{n+1} = f(y_n, x_n) \qquad \dots (2.2)$$

For all $n \geq 0$.
Now, we denote

$$\delta_n(t) = F_{gx_n,gx_{n+1}}(t) * F_{gy_n,gy_{n+1}}(t)$$

From (2.2) and (2.1) we get

$$F_{gx_n gx_{n+1}}(kt) = F_{f(x_{n-1},y_{n-1})f(x_n,y_n)}(kt) \geq F_{gx_{n-1}gx_n}(t) * F_{gy_{n-1}gy_n}(t) = \delta_{n-1}(t) \qquad \dots (2.3)$$

Similarly from (2.1) and (2.2)

$$F_{gy_n gy_{n+1}}(kt) = F_{f(y_n,x_n)f(y_{n-1},x_{n-1})}(kt) \geq F_{gy_{n-1}gy_n}(t) * F_{gx_{n-1}gx_n}(t) = \delta_{n-1}(t) \qquad \dots (2.4)$$

Adding by t-norm $*$ (2.3) and (2.4) we obtain

$$\delta_n(kt) \geq \delta_{n-1}(t) * \delta_{n-1}(t) \geq [\delta_{n-1}(t)]^2 \quad \dots \quad (2.5)$$

Thus we have

$$\delta_n(t) \geq \left[\delta_n\left(\frac{t}{k}\right)\right]^2 \geq \cdots \left[\delta_0\left(\frac{t}{k^n}\right)\right]^{2n}$$

That is, we have

$$F_{gx_n gx_{n+1}}(kt) * F_{gy_n gy_{n+1}}(kt) \geq \left[F_{gx_0 gx_1}\left(\frac{t}{k^n}\right)\right]^{2n} * \left[F_{gy_0 gy_1}\left(\frac{t}{k^n}\right)\right]^{2n} \dots (2.6)$$

On the other hand, since

$$t(1-k)(1+k+\cdots \dots .+k^{m-n-1}) < t$$

For every $m > n$, and $0 < k < 1$ we have

$$F_{gx_n gx_m}(t) * F_{gy_n gy_m}(t) \geq F_{gx_n gx_{n+1}}\left(t(1-k)(1+k+\cdots \dots .+k^{m-n-1})\right) * F_{gy_n gy_{n+1}}\left(t(1-k)(1+k+\cdots \dots .+k^{m-n-1})\right)$$
$$\geq F_{gx_n gx_{n+1}}\left(t(1-k)\right) * F_{gy_n gy_{n+1}}\left(t(1-k)\right) * F_{gx_{n+1}gx_{n+2}}(t(1-k)k) * F_{gy_{n+1}gy_{n+2}}(t(1-kk*$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$* F_{gx_{m-1}gx_m}(t(1-k)k^{m-n-1}) * F_{gy_{m-1}gy_m}(t(1-k)k^{m-n-1})$$
$$\geq F_{gx_0 gx_1}\left((1-k)\frac{t}{k^n}\right) * F_{gy_0 gy_1}\left((1-k)\frac{t}{k^n}\right)$$

…………………………….

$$F_{gx_0gx_1}\left((1-k)\frac{t}{k^n}\right) * F_{gy_0gy_1}\left((1-k)\frac{t}{k^n}\right)$$
$$\geq \left[F_{gx_0gx_1}\left((1-k)\frac{t}{k^n}\right)\right]^{m-n}\left[F_{gy_0gy_1}\left((1-k)\frac{t}{k^n}\right)\right]^{m-n}$$

$$\geq \left[F_{gx_0gx_1}\left((1-k)\frac{t}{k^n}\right)\right]^{m}\left[F_{gy_0gy_1}\left((1-k)\frac{t}{k^n}\right)\right]^{m}$$
$$\geq \left[F_{gx_0gx_1}\left((1-k)\frac{t}{k^n}\right)\right]^{n^q}\left[F_{gy_0gy_1}\left((1-k)\frac{t}{k^n}\right)\right]^{n^q} \to 1 \quad .$$

Where $q > 0$ such that $m \leq n^q$. therefore for each $0 < \varepsilon < 1$ and $t > 0$ there exist $n_0 \in \mathbb{N}$ such that for each $m, n \geq n_0$

$$F_{gx_n gx_m}(t) * F_{gy_n gy_m}(t) > 1 - \varepsilon$$

hence

$$F_{gx_n gx_m}(t) > 1 - \varepsilon$$

and

$$F_{gy_n gy_m}(t) > 1 - \varepsilon \,.$$

This shows that $\{gx_n\}$ and $\{gx_m\}$ are Cauchy sequences in X.
Since X is complete there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \text{ and } \lim_{n\to\infty} gy_n = y \quad \ldots \quad (2.7)$$

From (2.7) and continuity of g we have

$$\lim_{n\to\infty} ggx_n = gx \text{ and } \lim_{n\to\infty} ggy_n = gy \quad \ldots \quad (2.8)$$

We now show that

$$gx = f(x,y) \text{ and } gy = f(y,x)$$

By the semi-compatibility of $[g, f]$ we get

$$\lim_{n\to\infty} F_{gf(x_n,y_n)f(x,y)}(t) = \lim_{n\to\infty} F_{ggx_{n+1}f(x,y)}(kt) = 1$$

This yields

$$F_{gxf(x,y)}(kt) = 1 \,, \qquad \text{(by using (2.8))}$$

Hence

$$gx = f(x,y)$$

Similarly we can show that

$$gy = f(y,x)$$

Now we show that $gx = y$, and $gy = x$.
From (2.2) and (2.1) we get

$$F_{gxg\,y_{n+1}}(kt) = F_{f(x,y)f(y_n,x_n)}(kt) \geq F_{gxg\,y_n}(t) * F_{gyg\,x_n}(t)$$

So letting $n \to \infty$ this yields

$$F_{gx\,y}(kt) \geq F_{gx\,y}(t) * F_{gy\,x}(t) \quad \ldots \quad (2.9)$$

Similarly from (2.2) and (2.1)

$$F_{gx_{n+1}gy}(kt) = F_{f(x_n,y_n)f(y,x)}(kt) \geq F_{gx_n gy}(t) * F_{gy_n gx}(t)$$

So letting $n \to \infty$ this yields

$$F_{gy\,x}(kt) \geq F_{gy\,x}(t) * F_{gx\,y}(t) \quad \ldots \quad (2.10)$$

Adding by t –norm $*$ (2.9) and (2.10) we obtain

$$F_{gx\,y}(kt) * F_{gy\,x}(kt) \geq F_{gx\,y}(t) * F_{gy\,x}(t) * F_{gy\,x}(t) * F_{gx\,y}(t)$$
$$\geq \left[F_{gx\,y}(t) * F_{gy\,x}(t)\right]^2$$

Thus

$$F_{gx\,y}(t) * F_{gy\,x}(t) \geq \left[F_{gx\,y}\left(\frac{t}{k}\right) * F_{gy\,x}\left(\frac{t}{k}\right)\right]^2 \geq \left[F_{gx\,y}\left(\frac{t}{k^2}\right) * F_{gy\,x}\left(\frac{t}{k^2}\right)\right]^4$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\geq \left[F_{gx\,y}\left(\frac{t}{k^n}\right) * F_{gy\,x}\left(\frac{t}{k^n}\right)\right]^{2n}$$
$$\geq \left[F_{gx\,y}\left(\frac{t}{k^n}\right)\right]^{2n}\left[F_{gx\,y}\left(\frac{t}{k^n}\right)\right]^{2n} \to 1 \text{ as } n \to \infty.$$

Hence

$$y = gx = f(x, y)$$

Similarly

$$x = gy = f(y, x)$$

Now we prove that $x = y$. Suppose to the contrary from (2.2) and (2.1) we get

$$F_{gx_{n+1}gy_{n+1}}(kt) = F_{f(x_n,y_n)f(y_n,x_n)}(kt) \geq F_{gx_n gy_n}(t) * F_{gy_n gx_n}(t)$$

So letting $n \to \infty$ this yields

$$F_{x\,y}(kt) \geq F_{x\,y}(t) * F_{y\,x}(t)$$

Thus

$$F_{x\,y}(t) \geq \left[F_{x\,y}\left(\frac{t}{k}\right)\right]^2$$

$$\geq \left[F_{x\,y}\left(\frac{t}{k^2}\right)\right]^4$$

$$\dots$$

$$\geq \left[F_{x\,y}\left(\frac{t}{k^n}\right)\right]^{2n} \to 1 \text{ as } n \to \infty.$$

Thus we prove that

$$x = gx = f(x, x).$$

That is f and g have a common fixed point. The uniqueness of x follows from inequality (2.1).

Corollary 3.2 Let $a * b \geq ab$, for all $a, b \in [0,1]$ and $(X, F, *)$ be a complete Menger space such that F has n-property. Let $f: X \times X \to X$ be a functions such that

$$F_{f(x,y)f(u,v)}(kt) \geq F_{xu}(t) * F_{yv}(t)$$

for all $x, y, u, v \in X$, where $0 < k < 1$,

Then there exists a unique $x \in X$, such that $= f(x, x)$.

Proof - If we set $g = I$ in theorem 3.1 then the proof is complete .

Example 3.3 Let $X = [-2, 2]$, $a * b = ab$ for all $a, b \in [0,1]$ and $\varphi(t) = \frac{t}{t+1}$.

Then $(X, F, *)$ is a complete Menger space where

$$F_{xy}(t) = [\varphi(t)]^{|x-y|}$$

For all $x, y \in X$. define map f on $X \times X$ as follows

$f(x, y) = \frac{x^2}{8} + \frac{y^2}{8} - 2$ and $g(x) = x$.

It is easy to see that $f(X \times X) = [-2, -1]$ therefore $f(X \times X) \subseteq g(X)$

Then f satisfies all the conditions of theorem 3.1 and there exist a point.

$x = 2 - 2\sqrt{3}$, which is the unique common fixed point of f and g.

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