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Introduction of Inverse Problem and Its Applications to Science and Technology

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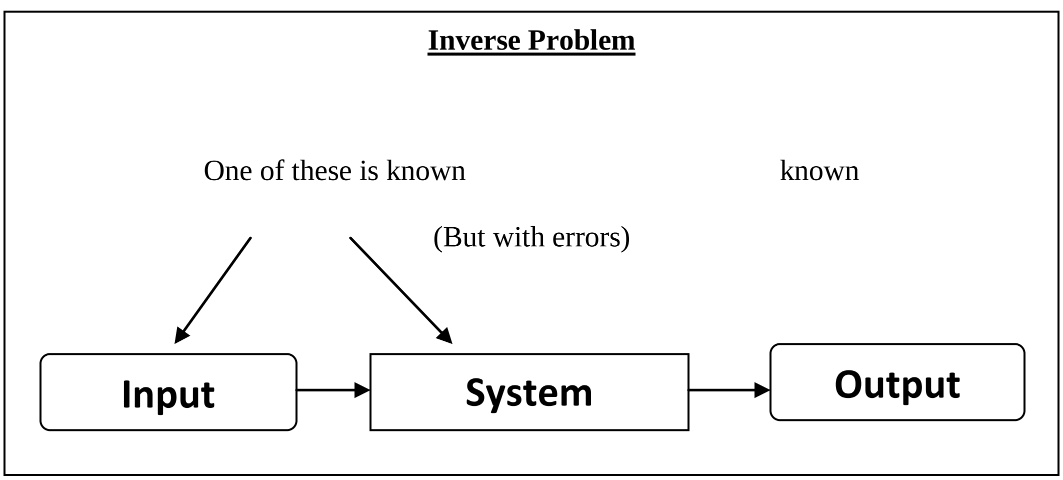
Abstract- Inverse problem is to deduce cause from effect. A wide variety of scientific problems end with the situation where if m is the desirable quantity, instead of m we have G(m) accessible for some operator G. Inverse problem is to find out m given G(m). In this article we will introduce the brief concept of Inverse problem. Besides carefully defining the problem, we give some mathematical examples also to fully understand the concept and the difficulties associated with the study of these problems. We will also point out some of the applications of these problems in the field of science and technology.

Keywords- Inverse problem, Ill- posed problems, Existence, Uniqueness, Continuous stability.

1. Introduction

Inverse problem is a vibrant and expanding branch of Mathematics that has found numerous applications. Two problems are called inverse to each other if the formulation of each of them requires full or partial knowledge of the other. Inverse problems deals with determining, for a given input-output system, an input that produces an observed output, or determining an input that produces a desired output (or comes as close to it as possible), often in the presence of noise.



In the fig. 1.1 the forward problem is to compute the output, given a system and the input to the system. The inverse problem is to compute either the input or the system, given the other two quantities. In most situations we have imprecise (noisy) measurements of the output. Now it is obviously arbitrary which of the two problems we call the direct and which we call the inverse problem. Hadamard introduced the concept of a well-posed problem, originally from the philosophy .

Jaque Hadamard (1865-1963)

a legendary mathematician of his time called a problem **well-posed** if the following conditions are satisfied

- Existence of solution
- Uniqueness
- Continuous Dependence-Stability, that is, small change is the cause (input) will make small change is the effect (output).

A problem violating any are of these conditions is called **Ill-posed** Inverse problems. It turns out that many interesting and important inverse problems in science lead to ill defined problem whereas the corresponding direct problems are well posed. Often existence and uniqueness can be forced by enlarging or reducing the solution space (the space of Models). For restoring stability however, one has to change the topology of space, which is in many cases impossible because of presence of measurement errors. At first glance it seems to be impossible to compute the solution of the problem numerically, if the solution of a problem does not depend continuously on the data *i.e.* for the case of ill-posed problem. Under additional a priori information about the solution such as smoothness and bounds on the derivative, however it is possible to restore stability and construct efficient numerical algorithms. It is often required to relate physical parameter '*x*' that characterizes a model to acquire observations making up some set of data '*y'*. Having a clear understanding of the underlying model, an operator can be specified relating *x* to *y* through the equation:-

$$F(x) = y$$

formulated in same appropriate vector space setting. The problem of estimating *x* from a measurement of *y* is a prototype of an inverse problem.

If the operator F is linear, the inverse problem is termed to be linear; otherwise it is a non-linear inverse problem. It turns out that non-linear inverse problems are considerably harder to solve than the linear ones. One crucial aspect of data collection is that the data *y* is often corrupted by some amount of noise.

We consider next the notion of a well- posed problem

$$A: x \in X \longrightarrow Ax = y \in Y. \qquad (1)$$

The problem (1) is said to be well-posed if for each "data" y in the data space Y, the above equation (1) has one and only one solution, and the Solution depends continuously on y. A well- posed problem is also known as a properly – posed problem.

In common man's language inverse problems can be described as follows:

- We want to guess question, when answer is known.

For example:

Answer: The capital of India is Delhi.

What is the question?

Question: What is the capital of India?

- We hear the music of a drum. Can we find the shape of the drum after having the music (by knowing frequencies)?
- More precisely *Marc Kac* (1966) posed the question "Can you deduce the shape of a drum?"
- In other words "can you deduce the shape of a plane region by knowing the frequencies at which it resonates (Boundary is fixed as in real drum)?
- Finding location of a ship by hearing its whistle could be an inverse problem

Inverse problems are associated with practically every field of human knowledge and mathematical equations such as matrix equations, ordinary differential equations, partial differential equations, integral equations.

Some mathematical examples are:

Consider the inhomogeneous Helmholtz equation

$$(\Delta + k^2 n^2)u = f \ in\ \Omega,) \text{ u = f in } \Omega,$$

Where Ω is a domain in R^3 and $k > 0$ is given.

The inverse problem here is to identify the coefficient n(x), the index of refraction.

A similar model BVP is the following

$$-\nabla. + (a\nabla u)\ ku = f \quad in\ \Omega,$$

where Ω is open bounded domain and the constant k is known. One instance of the appearance of the above BVP is the standing waves on a bounded shallow body of water with

$$k = \frac{4\pi^2}{gT^2}$$

where a is the water depth at the quiescent state, u is the elevation of the free surface above the quiescent level, g acceleration of gravity and T period of oscillation. The above equation then holds under some simplifications and Neumann boundary conditions augment it. *The direct problem in this setting is to find u. On the other hand, the corresponding inverse problem is to find a, given some measurement of u.*

Similarly,

The *Fredholm integral equation of the first kind* takes the generic form

$$\int_0^1 K(s,t)f(t)dt = g(s), \qquad 0 \le s \le 1,$$

Where the function K, is given by

$$K(s, t) = \frac{d}{(d^2 + (s-t)^2)^{3/2}}$$

Here, both the kernel K and the right hand side g are known functions, while f is the unknown function. This equation establishes a linear relationship between the two functions f and g and the kernel K describes the precise relationship between the two quantities. Thus the function K describes the underlying model.

If f and K are known, then we can compute g by evaluating the integral; this is called the forward computation. The inverse problem consists of computing f given the right hand side and the kernel.

Now from the above examples it is very clear to understand the concept of inverse problems.

2. Applications of Inverse Problems in the Field of Science and Technology

Inverse problems arise in many branches of science and mathematics, including: computer vision, machine learning, statistics, statistical inference, geophysics, medical imaging, remote sensing, ocean acoustic tomography, nondestructive testing, astronomy, physics, combinatorial chemistry, aerospace and many other fields.

A. *Imaging*

The forward problem is the mapping from the image to the quantities that we are able to measure. Details of the forward problem are given by some physical theory.

Thus the mapping from the image to the actual data is given by the relation

$$Y = A(x) + n$$

The inverse problem is then the finding the original image given the data and knowledge of the forward problem.

For example, in the case of deblurring photographs, the 'image' is the sharp photograph, the data is the blurred photograph, and the forward problem is the blurring process. The inverse problem is to find the sharp photograph (image) from the blurred photograph (data) and knowledge of the blurring process.

B. *Elasticity Imaging*

Elasticity imaging is a relatively new and promising technique in medical imaging. It relies on using the difference in elastic modules of tissues to distinguish them. In the forward problem, we are given the material properties and boundary data and are asked to calculate the displacement field. In the inverse problem the situation is reversed. We are now given the displacement field (or a related measurement) and the boundary data and are asked to calculate the material properties. This situation is typical of inverse problems in other fields such as ultrasound tomography and seismic imaging.

C. *Bio-mechanical imaging*

Biomechanical imaging has made it possible to image mechanical properties of tissues. These include properties such as the Young's modulus, nonlinear elastic properties and viscoelastic and poroelastic properties. Simultaneously, it has been established that the mechanical properties of tissue are altered if it is diseased.

Almost all techniques in biomechanical imaging involve watching the tissue deform using a standard imaging modality such as ultrasound or MRI, and then using the pre and post- deformation images to estimate the interior displacement of tissue. This displacement data is then used along with an appropriate mechanical model (say linear elastic) to determine the spatial distribution of the desired mechanical property.

D. *Geophysics*

Inverse problems have always played an important role in geophysics as the interior of the earth is not directly observable yet the surface manifestation of waves that propagate through its interior are measurable. Using the measurements of seismic waves to determine the location of an earthquake's epicenter, or the density of the rock through which the waves propagate, are typical of inverse problems in which wave propagation is used to probe an object.

E. *Astronomy*

Astronomy is by its very essence an inverse problem and can be regarded as the ultimate one, both in scale and difficulty, in the sense that astronomers are attempting to determine the structure of the universe from data arriving at one point in it. Apart from space exploration of the solar system, all cosmic objects are so remote that knowledge of them can only be inferred from remotely sensed data. There is no possibility of active experimentation as opposed to passive observation nor of obtaining stereoscopic information by change of viewpoint. Since the source cannot be actively changed the only way to increase the recorded signal (and so reduce noise) is to build larger collectors. As in all experimental work, the data recorded have to be deconvolved through an instrument response function to yield the arriving signal.

F. *Combinatorial Chemistry*

The inverse problem for topological indices (the σ- index, the c- index, the Z- index and Wiener index) in combinatorial chemistry are closely related to the design of combinatorial libraries for drug discovery. These indices are very popular in combinatorial chemistry. A topological index is a map from the set of chemical compounds represented by graphs to the set of real numbers. Experimental results show that many topological indices are closely correlated with some physicochemical characteristics. The inverse problem is defined as follows: given an index value, one wants to design chemical compounds (given as graph or trees) having that index value. The inverse problem has applications in the design of combinatorial libraries for drug discovery.

G. *Physics*

There are several inverse problems which have arisen in physics:

1) *Inverse moment problem*: Moments problems deal with recovery of a function or signal from its moments and the construction of efficient stable algorithms for determining or approximating the function.
2) *Inverse eigen value problem:* In quantum mechanics, the wave function of a mass in a potential satisfies the Schrodinger equation. The inverse problem is that of finding the potential given the eigen values.
3) *Inverse scattering problem:* Suppose a moving particle is repelled from a fixed scattering centre by a force derivable from a potential. Given the differential scattering cross section and energy of particle, one can find the potential.
4) *Determination of the shape of a hill from travel time:* suppose we slide a particle up a frictionless hill with initial energy and measure the time required for it to return. If we vary the initial energy and measure the time required to return, we can determine the shape of the hill.
5) *Determination of a potential from the period of oscillation:* Consider the motion of a particle in potential well. Given the period of oscillations of the particle with energy in a potential well, we can determine the potential.

H. *Signal Processing*

Given the result of passing a signal through a medium which acts as a filter, how can we reconstruct the original signal before the filtering occurred? For example, telephone lines will distort signals passing through them, and it is necessary to compensate for these distortions to recover the original signal. The problem of characterizing a linear, shift invariant system by determining its impulse response is usually a problem in deconvolution.

I. *Psychological science*

Perception can be viewed as an inverse problem that depends critically on the operation of a priori constraints.

Consider a mapping A from the distal stimulus X (e.g. a 3 D object) to the proximal stimulus Y (e.g. its retinal image). If the object and its image are presented by homogeneous co-ordinates, the perspective mapping A is a linear transformation(Y = AX).

Finding the proximal stimulus for a given distal stimulus is a direct problem. The inverse problem is: giving the proximal stimulus, perception is about inferring the properties of the distal stimulus.

J. *Economics*

In economic theory of demands, the direct problem is: given an exchange economy consisting of K consumers, there is an associated collective demand function, which is the sum of the individual demand functions. It maps the price system p to a goods bundle x (p). The inverse problem is: given a map $p \rightarrow x(p)$, it is natural to ask whether it is the collective demand function of a market economy.

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