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Disconnected and Connected Spaced

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Abstract:

We have discussed the basic properties of connected spaces regarding subspaces, product spaces, preservation under mappings etc. Also we have given several characterizations of these spaces. We begin our research paper of topological properties by making the idea of being connected that is being in one piece. It turns out to be easier to think about the property that is opposite of connectedness, namely the property of being in two or more pieces.

Keywords: Topological, Connected, Connectedness.

I. INTRODUCTION

Two important and interrelated strands in the practice of the exact sciences in the 19th century will be considered in which topological ideas came to be relevant for natural philosophy. In this way, light can be thrown on a part of the causal weave of events that eventually led to the emergence-of topology as a discipline, a part which has largely been neglected up until now in-the historical literature. The first of these two strands was concerned with topological issues that arose in the context of a dynamical theory of physical phenomena, a theory-advocated in particular by British natural philosophers during the last third of the 19th century. These developments will be discussed in the first part of our study. The second strand of events related to speculations about the large-scale topological structure of space will be the focus of the second part of this article.

The emergence of an entirely new discipline within mathematics is a rare event-in the history of science. The creation of topology the science of properties of spaces-and figures that remain unchanged under continuous deformations represents a phenomenon of this kind, but of a distinctly modern variety. Topology bears comparison-with the calculus, probability theory or number theory in that the first ideas about a new field called Analysis Situs or Geometria Situs were communicated among a handful of mathematically minded intellectuals in the late seventeenth and early eighteenth centuries. However, unlike the calculus and number theory, but similar to probability theory, the basic ideas underlying Analysis Situs reveal no ancient roots. Notoriously, ancient-authors treated questions of continuity hardly at all, and if so, then mainly as physical-questions linked to the phenomenon of motion. Moreover, in sharp contrast to these three other fields, during the 18th century no clearly defined domain of mathematical-problems was delineated that should and could be treated by Analysis Situs. Rather, a vague idea about an analysis which dealt not with magnitude, but "position," left it to-individual mathematicians to decide what should belong to the new field. Only gradually over the course of the 19th century was a consensus reached about the nature of-problems in topology. Nevertheless, after crossing the threshold to a scientific discipline in the full sense of the word in the first decades of this century, topology became one of the core research fields of mathematics, and topological arguments have come to play a role in virtually every other field in mathematics and the mathematical sciences. If one may reasonably speak of genuinely modern mathematical disciplines, then topology-certainly belongs among them. These late beginnings may be one reason why the emergence of topology has only begun to attract historiographical attention comparable to that received by fields like the calculus, number theory, or probability theory. While the invention of the calculus has long since been the object of historical study, and while the emergence of number theory and probability theory have recently been treated from a wide variety of perspectives, the number of historical monographs devoted to the formation of topology remains very small. Apart from these, we have a few survey articles and several research papers-dealing with particular topics within or closely related to topology.

II. REVIEW OF LITERATURE

Connectedness plays a very significant role in the study of topological spaces. The first attempt to give a precise definition of these spaces was made by Weierstrass who infact introduced the notion of arcwise connectedness. However, the notion of connectedness which we use today was introduced by Cantor (1883). Since then a host of leading toplogists notably Jordan (1893), Schoenfliesz (1902), F. Riesz (1906), Lennes (1911), Mazurkiewicz (1920), Vaidyanathaswamy (1947) studies these spaces very extensively and also introduced various generalization too of these spaces.

III. RESULTS & DISCUSSIONS

The only sets which can not be broken into pieces are the empty set and a set consisting of exactly one element. Showing that X is broken in two pieces which are disjoint Now let us consider the interval I = [0, 1] and the set X = [0, 1] ∪ [2, 3] , X has two pieces namely [0, 1] and [2, 3] but I has no such piece.

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(1) Definition (RO) : Let X be a topological space and let A and B be any two nonempty sets in X. Then A and B are said to be separated if :

$A \bigcap \overline{B} = \phi$ and $\overline{A} \bigcap B = \phi$

or $A \bigcap \overline{B} = \phi$ and $\overline{A} \bigcap B = \phi$

(2) Example : Let R be the set of real numbers with the usual topology. Consider the sets

A =]1, 2[, B =] 2, 3 [and C = [2, 3[

then clearly $A \bigcap \overline{B} =]1,\ 2[\bigcap [2,3] = \phi$

and $\overline{A} \bigcap B = [1,\ 2] \bigcap]2,3[= \phi$

Showing that A and B are separated. However, since

$\overline{A} \bigcap B = [1,\ 2] \bigcap]2,3[= \phi$

this shows that A and C are not separated.

(3) Definition (RO) : Let X be a topological space. By a separation of X we mean the existence of a pair of separated subsets of X whose union is X.

(4) Definition (RO) : Let X be a topological space then X is said to be disconnected if there exists two non-empty separated sets A and B such that

X = A ∪ B

(5) Definition (RO) : A topological space X is said to be connected if it is not disconnected.

(6) Example : We again consider the interval I = [0, 1] and the set

X = [0, 1] ∪ [2, 3]

Here X is disconnected since there exist a separation of X into [0, 1] and [2, 3] which are non-empty and disjoint. So X is not connected. While I = [0, 1] has no such separation. So I is not disconnected.

Hence I = [0, 1] is connected.

(7) Theorem (RO) : Let X be a topological space then X is disconnected iff X has a non empty proper subset which is both open and closed.

Proof : Let A be a non-empty proper subset of X which is both open and closed. Then (X – A) is also a non-empty proper subset of X which is both open and closed.

So X is the union of two non-empty separated sets, showing that X is disconnected.

Conversely : Let X be a disconnected space then there exists two nonempty separated sets A and B such that

X □ □ AUB

Since A and B are separated, therefore $A \bigcap \overline{B} = \phi$ and c

So $\quad \overline{A} \bigcap B = X,\ A \bigcap \overline{B} = X$ and $A \bigcap B = \phi$ (i)

Now $\quad A \bigcup B = X,\ A \bigcap B = \phi \Rightarrow A = X - B$ (ii)

and $\quad A \bigcup \overline{B} = X,\ A \bigcap \overline{B} = \phi \Rightarrow A = X - \overline{B}$ (iii)

also $\quad \overline{A} \bigcup B = X,\ \overline{A} \bigcap B = \phi \Rightarrow B = X - \overline{A}$ (iv)

Since $A \neq \phi,\ B \neq \phi$, it follows from (ii) that A is a non-empty proper subset of X and (iii) shows that A is open. (ii) and (iv) both shows that A is closed. Thus X has a non-empty proper subset which is both open and closed.

(8) Theorem (RO) : Let X be a topological space then X is disconnected iff X□ □ A UB where A and B are non empty disjoint open sets.

Proof : Let X is disconnected then there exist a nonempty proper subset A of X which is both open and closed then X – A is also a non empty subset of X which is both open and closed.
This shows that X is the union of two non-empty disjoint open sets.
Conversely : Let X be the union of two non-empty disjoint open sets A and B, then X – B = A.
Since B is open this implies A is closed and since B ≠ ϕ this implies A is non empty proper subset of X that is both open and closed. Hence X is disconnected.
(9) Theorem (RO) : Let X be a topological space then X is disconnected iff X □ □ AUB where A and B are non empty disjoint closed sets.
Proof : Let X is disconnected then there exist a non empty proper subset A of X which is both open and closed and X – A is also a non-empty subset of X which is both open and closed this shows that X is union of two non empty disjoint closed set.
Conversely : Let X be the union of two non-empty disjoint closed sets A and B then X – B = A.
Since B is closed this implies A is open and since B ≠ ϕ this implies that A is a non empty proper subset of X that is both open and closed. Hence X is disconnected.
(10) Theorem (RO) : Let X be a topological space then X is connected iff the only subsets of X that are both open and closed in X are the empty set and X itself.
Proof : Let X is connected and let A be a non empty proper subset of X which is both open and closed in X. Then the sets A and X – A form a separation of X. Since they are disjoint and nonempty and their union is X. This gives that X is disconnected. Which is a contradiction.
Conversely : Let X be a disconnected space. Let A and B form a separation of X. Then A is non empty and different from X and it is both open and closed in X.
(11) Theorem (RO) : Let X be a topological space. Then X is connected iff one of the following condition hold :
(i) There does not exist a separation of X.
(ii) X can not be decomposed into two disjoint, non empty open sets.
(iii) There does not exist a proper non-empty subset of X which is both open and closed in X.
Proof : Follows from the definition (4), (5) and theorem (10).
(12) Example : Every indiscrete space is connected. Let X be an indiscrete space. Since empty set and X are the only subset of X which is both open and closed in X. So X is connected.
(13) Example : Every singleton set is connected.
Let X be a topological space and let x∈X. Then {x} can not be expressed as the union of two non-empty disjoint sets. So {x} has no separation and is therefore connected.
(14) Example : Every discrete space which contain more than one point is disconnected.
Let X be discrete space and let x∈X. Then {x} is a non empty proper subset of X which is both open and closed in X. Hence X is disconnected.
(15) Example : The rational Q are not connected.
If Y is a subspace of Q containing two points p and q we can choose an irrational number a lying between p and q such that Y can be written as the union of two disjoint open sets.
(16) Theorem (RO) : Let X be a topological space. Let Y be a subspace of X and let Y □ □ AUB wher A and B are non empty and disjoint sets neither of which contain limit point of other, is a separation of Y. Then the space Y is connected if there exists no separation of Y.
Proof : Let Y = AUB is a separation of Y. Then A is both open and closed in Y. Let $\overline{A}$ is the closure of A in X. Then closure of A in Y is the set $\overline{A} \cap Y$. Since A is closed in Y then A= $\overline{A} \cap Y$ or $\overline{A} \cap B = \phi$. Since A = $\overline{A} \cap Y$, where D (A) is set of all limit points of A. So B contains no limit point of A. In same way we can show that A contains no limit point of B.
Conversely : Let A and B are disjoint non-empty sets whose union is Y and neither of which contain a limit point of other then $\overline{A} \cap B = \phi$ and $\overline{A} \cap B = \phi$. So we have A = $\overline{A} \cap Y$ and B = $\overline{B} \cap Y$. Thus Thus both A and B are closed in Y and since A = Y – B and B = Y – A. So they are open in Y.
(17) Example : Consider the following subset of the plane R^2 where R^2 is enclosed with the product topology.

$\overline{A} \cap Y$ = A and $\overline{B} \cap Y$ =B

Then X is not connected. Since the two sets form a separation of X, because neither contain a limit point of other.
(18) Theorem (RO) : Let X be a topological space and let X = CUD

where C and D are non empty disjoint open sets in X. Let Y is a connected subset of X then either $Y \subset C$ or $Y \subset D$.

Proof : Since C and D are open in X. So the sets $C \bigcap Y$ and $D \bigcap Y$ are open in Y and since C and D are disjoint therefore $C \bigcap Y$ and $D \bigcap Y$ are disjoint and

$Y \subset C$ or $Y \subset D$.

if $C \bigcap Y \neq \phi$ and $D \bigcap Y \neq \phi$ then $C \bigcap Y$ and $D \bigcap Y$ form a separation of Y but Y is connected therefore either either $Y \bigcap C = \phi$ or $Y \bigcap D = \phi$

Hence either $Y \subset C$ or $Y \subset D$.

(19) Theorem (RO) : Let X be a topological space and let $A_\alpha \subset C \ \forall \alpha \in I$ collection of connected subsets of X with the property $A_\alpha \subset C \ \forall \alpha \in I$ is connected.

Proof : Let p be any point of $\bigcap A\alpha$. We prove that $A_\alpha \subset C \ \forall \alpha \in I$ is connected. Let $Y = C \cup D$ is a seperation of Y and p is in one of the sets C or D. Let $p \in C$. Since the set $A\alpha$ is connected so either $A \subset C$ or $A\alpha \subset D\alpha$, it cannot lie in D because it contain the point p of C.

Hence $A\alpha \subset C \ \forall \alpha \in I$

(20) Theorem (RO) : Let X be a topological space and let A be a connected set in X. If B is any subset of X such that $A \subset B \subset \overline{A}$ then B is also connected subset of X.

Proof : Let A be a connected set in X and let $A \subset B \subset \overline{A}$.

Now let $B = C \cup D$ is a separation of B. Since A is connected.

So A must lie either in C or in D.

Let $A \subset C$ then $\overline{A} \subset \overline{C}$, since $C \bigcup D = \phi$. So B can not intersect D.

contradiction gives that $D = \phi$ and hence B is a connected subset of X.

(21) Definition : Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a function from X into Y. Then f is said to be continuous if any of the following condition is satisfied :

(i) For each open subset V of Y, the set $f : X \rightarrow Y$ is an open subset of X.

(ii) For each closed subset B of Y, the set $f : X \rightarrow Y$ is closed in X.

(iii) For every subset A of X, we have $f : X \rightarrow Y$

(22) Theorem (RO) : Let $f : X \rightarrow Y$ be a continuous function from a connected space X into a topological space Y.

Then $f : X \rightarrow Y$ is connected.

Proof : Let $f : X \rightarrow Y$ be a continuous function and let X be connected.

We have to prove that $Z = f(X)$ is connected.

Since the function obtained from f by restricting its range to the space Z is also continuous. So we consider only the case of continuous subjective function

$g : X \rightarrow Z$

Let Z = AUB is a separation of Z into two disjoint non empty sets open in Z. Then $g : X \to Z$ are disjoint sets such that X = $g : X \to Z$

They are open in X because g is continuous and non empty as g is subjective, therefore they form a separation of X. This gives a contradiction to the fact that X is connected. Therefore f(X) is connected.

(23) Definition : Let $\Im$ and $\Im^*$ be two topologies on a given set X. If $\Im^* \supseteq \Im$ then $\Im$ is coarser than $\Im^*$.

(24) Theorem : Let X be a connected topological space with topology $\Im$ and $\Im^*$ is coarser than $\Im$. Then the space X with topology $\Im^*$ is also connected.

Proof : Let the space X with topology $\Im^*$ is disconnected then there exist a non empty proper subset A of X which is both open and closed, then A and X – A are both open in $\Im^*$, Since $\Im^* \subseteq \Im$, this implies that A and X– A are both open in $\Im$. This shows that A is a non empty proper subset of X which is both open and closed with respect to $\Im$. So space X with topology $\Im$ is disconnected which gives a contradiction. So space X with topology $\Im^*$ is connected.

(25) Theorem (RO) : (Connected sets in the real line). Let E be a subset of the real line R containing at least two points. Then E is connected iff E is an interval.

Proof : Let E be any subset of real line containing at least two points and let E is not an interval.

Let $a, b \in E$ and $p \notin E$ such that $a < p < b$.

Let

$A = E \bigcap G$ and $B = E \bigcap H$

then $a \in G$ and $b \in H$, therefore $E \bigcap G$ and $E \bigcap H$ are non empty disjoint sets whose union is E therefore E is disconnected.

Now let E is an interval and let E is disconnected.

Let G and H form a seperation of E and let $A = E \cap G$ and $B = E \cap H$, then $E = A \cup B$, where A and B are non empty sets. Let $a \in A$ and $b \in B$ such that $a < b$.

Let $p \in A = E \bigcap G$, since [a, b] is a closed set therefore $p \in [a, b]$.

Let $p \in A = E \bigcap G$ then $p < b$ and $p \in G$.

Since G is an open set therefore there exists $\delta > 0$ such that

$p + \delta \in G$ and $p + \delta < b$.

Hence $p + \delta \in E$, then $p + \delta \in A$.

This gives a contradiction to the definition of p therefore $p \notin A$.

Now let $p \in B = E \cap H$ then $p \in H$ and H is an open set therefore there exist $\delta^* > 0$ such that

$[p - \delta^*, p] \subset H$ and $a < p - \delta^*$

therefore $[p - \delta^*, p] \subset E$ and so $[p - \delta^*, p] \subset B$

Hence $[p - \delta^*, p] \bigcap A = \phi$ but then $p - \delta^*$ is an upper bound for $A \bigcap [a, b]$ which is not possible by definition of p therefore $p \notin B$. But this is a contradiction to the fact that $p \in E$. Hence E is connected.

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